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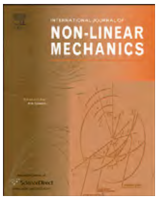

On the plastic bifurcation and post-bifurcation of axially compressed beams

Philippe Le Grognec [a],*, Anh Le van [b]

[a] *Ecole des Mines de Douai, Polymers and Composites Technology & Mechanical Engineering Department, 941 rue Charles Bourseul - BP 10838, 59508 Douai Cedex, France*
[b] *GeM (Laboratory of Civil and Mechanical Engineering), Faculty of Science - University of Nantes, 2 rue de la Houssinière - BP 92208, 44322 Nantes Cedex 3, France*

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ABSTRACT

The paper presents two new results in the domain of the elastoplastic buckling and post-buckling of beams under axial compression. (i) First, the tangent modulus critical load, the buckling mode and the initial slope of the bifurcated branch are given for a Timoshenko beam (with the transverse shear effects). The result is derived from the 3D J_2 flow plastic bifurcation theory with the von Mises yield criterion and a linear isotropic hardening. (ii) Second, use is made of a specific method in order to provide the asymptotic expansion of the post-critical branch for a Euler–Bernoulli beam, exhibiting one new non-linear fractional term. All the analytical results are validated by finite element computations.

1. Introduction

Failure of slender or thin structures is mainly due to the buckling phenomenon and necessitates the analysis of the buckling and post-buckling behaviors for their mechanical design, namely the calculation of the critical loads, the bifurcation modes and the post-critical equilibrium branches.

The buckling of a straight beam under axial compression is the most commonly studied in the literature, either in elasticity or in plasticity. The first critical load of a compressed beam in elasticity was found as the solution of an eigenvalue problem by Euler at the beginning of the eighteenth century. Much later on, in 1945, Koiter succeeded in describing the elastic post-bifurcation behavior using an asymptotic power expansion. At the same time, he also achieved the imperfection sensitivity analysis for an elastic buckled beam. On the other hand, the plastic bifurcation analysis, even on a simple model such as a beam, makes much less progress than in elasticity. It still is the subject of a lot of recent theoretical as well as experimental investigations.

In the sequel, we will briefly recall the main results available in the literature on the plastic bifurcation, with no attempt to make an exhaustive review (see, for instance, the survey paper of Sewell [1] for a general review on plastic buckling). The interested reader can find more details on the theoretical developments and more complete states-of-art in the quoted references below.

The pioneering works on the critical load of an elastoplastic beam under axial compression dated back to the end of the nineteenth century and were conducted by Engesser, Considère, and later by von Karman. Yet, the early results before the 1940s were not quite correct or properly justified. Some authors derived the tangent modulus critical load by discarding the unloading possibility in the structure; others obtained the reduced modulus critical load by assuming that the bifurcation takes place at constant load as in elasticity. The first significant result is due to Shanley [2], who provided a rational explanation for the plastic buckling of the so-called Shanley's column, which had been introduced by von Karman in elasticity. The considered model is a rigid rod with two degrees of freedom; it is supported by two elastoplastic springs at the bottom and subjected to an axial compressive force at the tip. This discrete model supposedly able to reproduce the behavior of a beam cross-section did lead to results which are qualitatively similar to those of a continuum structure under plastic buckling. Shanley thus provided a satisfactory answer to the value and the nature of the first critical load. In discrete or continuous structures, the first bifurcation occurs at the tangent modulus critical load, giving rise to an incipient unloaded zone and an increasing load during the initial post-bifurcation. Hill [3] extended these results to a 3D continuum by using the concept of "comparison elastic solid". He examined the uniqueness and stability criteria, and pointed out the difference between bifurcation and stability.

As regards the plasticity theory to be used for solving the plastic buckling problem, one can choose between the J_2 deformation and flow theories. These two theories may yield different critical values and they coexist as they have each their own advantages and drawbacks. The deformation theory, although it does not take into

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* Corresponding author. Tel.: +33 3 27 71 23 21; fax: +33 3 27 71 29 81.
E-mail addresses: philippe.le.grognec@mines-douai.fr (P. Le Grognec), anh.le-van@univ-nantes.fr (A. Le van).

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