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Geometrically exact beam theory without Euler angles

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ABSTRACT

In modeling highly flexible beams undergoing arbitrary rigid–elastic deformations, difficulties exist in describing large rotations using rotational variables, including three Euler angles, two Euler angles, one principal rotation angle plus three direction cosines of the principal rotation axis, four Euler parameters, three Rodrigues parameters, and three modified Rodrigues parameters. The main problem is that such rotational variables are either sequence-dependent and/or spatially discontinuous because they are not mechanics-based variables. Hence, they are not appropriate for use as nodal degrees of freedom in total-Lagrangian finite-element modeling. Moreover, it is difficult to apply boundary conditions on such discontinuous and/or sequence-dependent rotational variables. This paper presents a new geometrically exact beam theory that uses no rotation variables and has no singular points in the spatial domain. The theory fully accounts for geometric nonlinearities and initial curvatures by using Jaumann strains, exact coordinate transformations, and orthogonal virtual rotations. The derivations are presented in detail, fully nonlinear governing equations and boundary conditions are presented, a finite element formulation is included, and the corresponding governing equations for numerically exact analysis using a multiple shooting method is also derived. Numerical examples are used to illustrate the problems of using rotational variables and to demonstrate the accuracy of the proposed geometrically exact displacement-based beam theory.

1. Introduction

A flexible multibody system consists of interconnected rigid and deformable components and each component may undergo large translations and rotations (Shabana, 2005; Bauchau, 2010; Kane et al., 1983). Modeling and analysis of a flexible multibody system that undergoes large rotations is very challenging because geometric nonlinearities exist in flexible components and equations of motion of rigid components are nonlinear ordinary differential equations (Shabana, 2005; Bauchau, 2010). Hence, nonlinear finite-element modeling with iteration techniques is often used in the modeling and analysis of flexible multibody systems. Even with the use of finite elements, however, many challenging problems still exist, and the most challenging task is how to accurately describe large rotations of flexible and rigid components without singularity problems in the space and time domains. One way to reduce the coupling-induced complexity of governing equations is to derive and use total-Lagrangian structural theories referred directly to an inertial reference frame without using any floating reference frames (Kane et al., 1987). Because the strain–displacement relations of a total-Lagrangian structural theory fully account for both rigid and elastic deformations, there is no need of complicated, problem-dependent nonlinear terms to describe the coupling of rigid and flexible components. Moreover, total-Lagrangian nonlinear rotary inertial terms of a differential flexible component have the same form as those of a rigid body (Pai, 2007). However, challenging issues exist in the derivation and analysis of geometrically exact total-Lagrangian displacement-based structural theories.

An initially curved beam undergoing large rigid–elastic deformation requires three coordinate systems to describe its motion, as shown in Fig. 1(a). The *abc* is a fixed rectangular coordinate system used for reference, the *xyz* is a fixed orthogonal curvilinear coordinate system used to describe the undeformed beam geometry, and the $\xi\eta\zeta$ is a moving orthogonal curvilinear coordinate system used to describe the deformed beam geometry. Let $\mathbf{i}_a$, $\mathbf{i}_b$, and $\mathbf{i}_c$ be the unit vectors of the *abc* system; $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ be the unit vectors of the *xyz* system; and $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ be the unit vectors of the $\xi\eta\zeta$ system. Moreover, u, v, and w represent the absolute displacements of the observed reference point O with respect to (w.r.t.) the x, y, and z axes, respectively, and s denotes the undeformed arc length along the reference line starting from the beam root. Because u, v, and w are continuous functions of the spatial coordinate s (and the time t if a dynamic problem), $v'(\equiv \partial v/\partial s)$, w' and u' exist and they can exactly describe the reference line's bending rotations of any magnitude (Pai, 2007). However, a torsional angle ϕ (see, e.g., Fig. 1(a)) is still needed in order to describe the twisting of

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