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Eulerian formulation of constrained elastica

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ABSTRACT

The formulation of the constrained elastica problem proposed in this paper is predicated on two key concepts: first, the deformed elastica is described by means of the distance from the conduit axis; second, the problem is formulated in terms of the Eulerian curvilinear coordinate of the conduit rather than the natural curvilinear coordinate of the elastica. This approach is further implemented within a segmentation algorithm, which transforms the global constrained elastica problem into a sequence of analogous auxiliary problems that result from dividing the conduit and the elastica into segments limited by contacts. Each auxiliary segment entails solving a segment of elastica subject to isoperimetric constraints corresponding to the assumed positions of the segment ends along the conduit. This new formulation resolves in one stroke a series of issues that afflict the classical Lagrangian approach: (i) the contact detection is reduced to checking whether a threshold on the distance function is violated, (ii) the isoperimetric conditions are transformed into regular boundary conditions, instead of being treated as external integral constraints, (iii) the method yields a well-conditioned set of equations that does not degenerate with decreasing flexural rigidity of the elastica and/or decreasing clearance between the conduit and the elastica.

Nomenclature

Two distinct curvilinear coordinates are introduced in the formulation of the constrained elastica problem: the Lagrangian coordinate s along the elastica and the Eulerian coordinate S along the conduit axis. More generally, we adopt the convention of using lower case for quantities pertaining to the elastica and upper case for variables describing the conduit. Furthermore, field variables for the elastica can be viewed either as Lagrangian (s) or as Eulerian (S) functions; in the former case, the quantities are tagged with a tilde ($\sim$). Calligraphic symbols denote dimensionless forces, while Greek letters are used for dimensionless quantities. Finally, the inserted object is referred to as a "rod" for simplicity, despite the assumed planar nature of the problem.

1. Introduction

The insertion of a flexible rod into a curved hollow conduit is a basic problem encountered in many medical and engineering applications. Examples include the insertion of a guidewire into the artery of a patient as part of a procedure to deploy a stent to prop the artery wall (Chen and Li, 2007), the endoscopic examination of internal organs (Katopodes et al., 2001), the endoscopic investigation of pipe systems, and the insertion of artificial fibers in industrial crimpers (Cooke, 2000). Another application concerns the drilling of multiple curved wells from a single platform to reach a deep hydrocarbon reservoir. Here the problem further requires considering the evolution of the borehole (Downton, 2007; Detournay, 2010), which is itself conditioned by the interaction between the borehole and the drillstring (Cunha, 2004). Energy dissipation devices based on the multiple folding of a flexible member inside a rigid enclosure provide yet another example of the type of structures under investigation (Chai, 2006). In other applications, the contact takes place on one side only of the elastica, such as the insertion of a paper sheet into a toner (Soong and Choi, 1986) or the uplift buckling of textile fabrics or railway tracks (Fraser, 2003).

This class of problems is known as constrained elastica. These problems are strongly nonlinear, because of the nonpenetration constraint between the flexible rod and the walls of the conduit, a condition known as unilateral contact (Brogliato, 1999). In addition, any large deflections of the rod from a stress-free configuration require considerations of a geometrically nonlinear model of the rod that can introduce parasitic solutions with curling of the elastica (Arreaga et al., 2002); indeed, these solutions are generally not physically admissible within the context of a rod inserted inside a conduit. Furthermore, in a large majority of applications that fall within the class of constrained elastica, the inserted length of the elastica is *a priori* unknown. The formulation of such problems

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