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# Ellipsoidal anisotropy in linear elasticity: Approximation models and analytical solutions

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ABSTRACT

The concept of *ellipsoidal anisotropy*, first introduced in linear elasticity by Saint Venant, has reappeared in recent years in diverse applications from the phenomenological to micromechanical modeling of materials. In this concept, *indicator surfaces*, which represent the variation of some elastic parameters in different directions of the material, are ellipsoidal. This concept recovers different models according to the elastic parameters that have ellipsoidal *indicator surfaces*. An interesting feature of some models introduced by Saint Venant is the formation of analytical solutions for basic problems of linear elasticity. This paper has two main objectives. First, an accurate definition of the variety of anisotropy called *ellipsoidal* is provided, which corresponds to a family of materials that depends on 12 independent parameters, including varieties of orthotropic and non-orthotropic materials. An explicit *nondegenerate* Green function solution is established for these materials. Then, it is shown that the *ellipsoidal* model recovers a variety of phenomenological and theoretical models used in recent years, specifically for geomaterials and damaged or micro-cracked materials. These models can be used to approximate the elastic parameters of any anisotropic material with different fitting qualities. A method to optimize the parameters will be given.



## 1. Introduction

In recent years, much research has focused on modeling the elastic anisotropy of noncrystalline materials with simple models. A major focus has been determining adequate models with a reduced number of parameters, which have been constructed using various methods. Cowin and Mehrabadi (1987, 1995) defined classes of elastic materials based on the number and orientation of reflective symmetry planes. This approach was applied to model bone elasticity (Yang et al., 1999). Special cases of the fourth order elasticity tensors have also been considered; these cases can be expressed in terms of only one second order tensor. This model has been applied to the micro-mechanical approach of damaged and micro-cracked materials (Halm and Dragon, 1988; Dragon et al., 2000; Sevostianov and Kachanov, 2002, 2008). In other studies, the *indicator surface* properties of some "mono-directional" elastic parameters have been used to define simple forms of elastic anisotropy. A "mono-directional" parameter is a parameter like Young's modulus or the elastic coefficient defined by Eq. (1) that depends on the elasticity tensor $\mathbb{C}$ and only one material direction $\underline{n}$. Examples of mono-directional elastic parameters can be found in Pouya (2007a). The *indicator surface* of a mono-directional parameter is the surface defined by the equation $\underline{x}(\underline{n}) = r(\underline{n})\ \underline{n}$, where $\underline{n}$ scans the unit sphere and represents a direction in the material and $r(\underline{n})$ is the distance from $\underline{x}$ to the origin of coordinates that equals the value of the elastic parameter in the direction $\underline{n}$. The characterization of material anisotropy by the shape of the indicator surfaces was initiated in the early work of Saint Venant (1863), who first introduced the concept of the ellipsoidal anisotropy. Because material isotropy geometrically corresponds to the image of a sphere, anisotropies corresponding to the image of an ellipsoid have naturally been investigated. Saint Venant (1863) studied several elasticity models of this type, arguing the utility of these models in representing the elasticity of anisotropic amorphous materials. This included the study of materials for which the indicator surface of $\sqrt[4]{E(\underline{n})}$, where $E(\underline{n})$ is the Young's modulus in the material direction $\underline{n}$ (see Eq. (31)), is an ellipsoid.

However, the research of Saint Venant has previously been neglected in the literature, and the only evidence is a short quote in a book by Lekhnitskii (1963). Independently, the concept of ellipsoidal anisotropy has reappeared in recent years as a guideline for modeling the elasticity of *geomaterials*, such as soils, rocks, and concrete. The anisotropic character of these geomaterials is being accounted for more frequently in different applications, including geotechnical design or the study of seismic wave propagation,

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