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On numerical experiments for Cauchy problems of elliptic operators

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ABSTRACT

Over the last decade, there has been a considerable amount of new numerical methods being developed for solving the Cauchy problems of elliptic operators. In this paper, with some new classes of numerical experiments, we re-verify the conclusions in the review article [Wei T, Hon YC, Ling L. Method of fundamental solutions with regularization techniques for Cauchy problems of elliptic operators. Eng Anal Bound Elem 2007;31(4):373–85.] concerning the effectiveness of solving Cauchy problems with the method of fundamental solutions.

1. Introduction

Cauchy problems of elliptic operators are typically ill-posed problems whose solutions do not continuously depend on the input Cauchy data. However, these ill-posed problems play important roles in many science and engineering models such as steady-state inverse heat conduction [10], electro-cardiology [6], nondestructive testing [11], and so on. For the sake of numerical computations, a small perturbation or error in the input may lead to an enormous error in the numerical solution.

The Cauchy problem we consider is in the form of

$$\mathcal{L}u = 0, \quad \text{in } \Omega \subset \mathbb{R}^d,$$

$$\partial_\nu^{(k)} u = g_k, \quad \text{on } \Gamma, k \in \{0,1\}, \tag{1}$$

where $\mathcal{L}$ is a differential operator of elliptic type, Ω is a bounded simply connected domain in $\mathbb{R}^d$, $\Gamma \subsetneq \partial\Omega$ is the Cauchy boundary, and $\partial_\nu^{(k)}$ is the k-th order normal-derivative. The goal here is to determine a distribution function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ that satisfies (1) with the provided Cauchy data g_k, $k \in \{0,1\}$.

2. Redundancy in Cauchy data

For convenience, researchers usually pick exact solutions that satisfy the governing equation $\mathcal{L}u = 0$ in the whole space $\mathbb{R}^d$ in order to generate the Cauchy data g_k, $k \in \{0,1\}$. This Cauchy data is then used to verify different numerical methods for solving (1). *In some situations*, this approach may yield overdetermined test problems—the key message we want to deliver in this paper. Although it is not our aim to come up with a general mathematical theory about the redundancy in Cauchy data, below is a specific situation in which the Neumann boundary data is *not necessary* to guarantee unique solution in the Cauchy problem.

Consider a Cauchy problem of the Laplace operator in two-dimensions for simplicity. Suppose the *exact* Cauchy solution u^* of (1) is harmonic everywhere in $\mathbb{R}^2$. For simplicity, let us consider Ω being the unit circle and the Cauchy boundary Γ being the upper half. Since u^* is harmonic and, therefore, analytic, the Dirichlet data $g_0(\theta)$, $\theta \in I := [0,\pi]$, is real analytic. Now further assume that the Taylor expansion of u^* at the origin has a radius of convergence $R > 1$. Then the two (1D real) analytic functions, $u^*|_{r=1}(\theta)$ and $g_0(\theta)$, agree on I; by the unique continuation of analytic functions, they also agree on $[-\pi/2, 3\pi/2]$. Having Dirichlet boundary condition on the whole boundary $\partial\Omega$ yields the Dirichlet (forward) problem and (1) has unique solution without the Neumann boundary condition.

Note that in the above situation, having a unique solution does not imply that the solution process is stable. The Cauchy problem is still ill-posed and is highly sensitive to any noise in the Cauchy data. We observe that the test problems in some literatures, i.e.

In [5]: $u^* = y^3 - 3yx^2$,
In [9]: $u^* = xy$,
In [16]: $u^* = \exp(0.5x)\sin(0.5y)$ and $u^* = x + y$,
In [17]: $u^* = x^3 - 3xy^2 + \exp(2y)\sin(2x) - \exp(x)\cos(y)$,
In [19]: $u^* = 10y - 9$, and others in [1–4,13–15],

the tested Cauchy problems with globally harmonic solutions may also have *numerically redundant Cauchy data*.

Numerically, we can sometimes solve the Cauchy problem (1) *with only one boundary condition* on Γ. Figs. 1 and 2 show some numerical reconstructions when we apply the MFS directly (without regularization) to solve (1) with Dirichlet data only. We show only a subset of the collocation points in order to keep the figures easily readable. Figs. 1(a) and 1(b) are the reconstructed solutions in a unit circle and square, respectively, using Dirichlet data $g_0 = \exp(x)\cos(y) = u^*|_\Gamma$ (with $\Delta g_0 = 0$). It can be seen that the numerical solutions closely agree with the

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