A generalization of the Mertens' formula and analogue to the Wallis' product over primes

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Abstract

In this paper, we study the asymptotic expansion of the product $\prod_{p\leqslant x}(1+\frac{\alpha}{p})$ for each fixed real $\alpha > -2$ where the p runs over the prime numbers. As an application, we study the Wallis' product and its generalizations, running over primes p, which are analogue to Wallis product for $\frac{\pi}{2}$ running over positive integers.

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1 Introduction

A generalization of the Mertens formula. Among his interesting three results in number theory related to the density of the primes, Mertens [2] proved a result asserting, in todays notation, that

$$\prod_{p\leqslant x}\Big(1-\frac{1}{p}\Big)=\frac{e^{-\gamma}}{\log x}\Big(1+O\Big(\frac{1}{\log x}\Big)\Big) \tag{1}$$

where the product runs over primes and γ denotes the Eulers constant. Several generalizations, and also improvements on the O-term in the above formula are obtained [3]. In this note we study the following generalization.

Theorem 1.1. *Assume that $\alpha > -2$ and $\alpha \neq 0$ is a fixed real, and define the constant $C(\alpha)$ by*

$$C(\alpha)=e^{\alpha\gamma}\prod_{p}\Big(1-\frac{1}{p}\Big)^{\alpha}\Big(1+\frac{\alpha}{p}\Big). \tag{2}$$

Then for each $x > 1$ we have

$$\prod_{p\leqslant x}\Big(1+\frac{\alpha}{p}\Big)=C(\alpha)(\log x)^{\alpha}\Big(1+O\Big(\frac{1}{\log^2 x}\Big)\Big)$$

Moreover, if we assume that the Riemann Hypothesis is true, then one may reduce the above O-term up to $O(x^{\frac{1}{2}}\log x)$.

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