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On a subalgebra of $C(X)$ containing $C_c(X)$

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Abstract

Let $C_c(X) = \{f \in C(X) : |f(X)| \leq \aleph_0\}$, $C^F(X) = \{f \in C(X) : |f(X)| < \infty\}$, and $L_c(X) = \{f \in C(X) : \overline{C_f} = X\}$, where C_f is the union of all open subsets $U \subseteq X$ such that $|f(U)| \leq \aleph_0$, and $C_F(X)$ be the socle of $C(X)$ (i.e., the sum of minimal ideals of $C(X)$). It is shown that if X is a locally compact space, then $L_c(X) = C(X)$ if and only if X is locally scattered. We observe that $L_c(X)$ enjoys most of the important properties which are shared by $C(X)$ and $C_c(X)$.

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1 Introduction

$C(X)$ denotes the ring of all real valued continuous functions on a topological space X. In [4] and [5], $C_c(X)$, the subalgebra of $C(X)$, consisting of functions with countable image are introduced and studied. It turns out that $C_c(X)$, although not isomorphic to any $C(Y)$ in general, enjoys most of the important properties of $C(X)$. This subalgebra has recently received some attention, see [4], [1], and [5]. Since $C_c(X)$ is the largest subring of $C(X)$ whose elements have countable image, this motivates us to consider a natural subring of $C(X)$, namely $L_c(X)$, which lies between $C_c(X)$ and $C(X)$. Our aim in this article, similarly to the main objective of working in the context of $C(X)$, is to investigate the relations between topological properties of X and the algebraic properties of $L_c(X)$. In particular, we are interested in finding topological spaces X for which $L_c(X) = C(X)$. An outline of this paper is as follows: We show that if X is a locally compact space, then $L_c(X) = C(X)$ if and only if X is locally scattered, which is somewhat similar to a classical result due to Rudin in [10], and Pelczynski and Semadeni in [8] (of course, by no means as significant). This classical result says that a compact space X is scattered if and only if $C(X) = C_c(X)$. Let us for the sake of the brevity, call the latter classical result, RPS-Theorem. If X is an almost discrete space or a P-space, then $L_1(X) = L_F(X) = L_c(X) = C(X)$, where $L_F(X)$ and $L_1(X)$ are the locally functionally finite (resp., constant) subalgebra of $C(X)$, see Definition 2.3.

All topological spaces that appear in this article are assumed to be infinite completely regular Hausdorff, unless otherwise mentioned. For undefined terms and notations the reader is referred to [6], [3].

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