An extension of $C_F(X)$

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Abstract

Let $C_F(X)$ be the socle of $C(X)$ (i.e., the sum of minimal ideals of $C(X)$). We define $LC_F(X) = \left\{f \in C(X) : \overline{S_f} = X\right\}$, where S_f is the union of all open subsets U in X such that $|U\backslash Z(f)| < \infty$, $LC_F(X)$ is called the locally socle of $C(X)$ and it is a z-ideal of $C(X)$ containing $C_F(X)$. We characterize spaces X for which the equality in the relation $C_F(X) \subseteq LC_F(X) \subseteq C(X)$ is hold. We determine the conditions such that $LC_F(X)$ is not prime in any subrings of $C(X)$ which contains the idempotents of X. We investigate the primness of $LC_F(X)$ in some subrings of $C(X)$.

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1 Introduction

$C(X)$ denotes the ring of all real valued continuous functions on a topological space X. We recall that a nonzero ideal E in a commutative ring R is called essential if it intersects every nonzero ideal nontrivially. Let I be an ideal in $C(X)$, then $Z[I] = \{Z(f) : f \in I\}$ and $Z(X) = \{Z(f) : f \in C(X)\}$. If $Z^{-1}[Z[I]] = I$, then I is called a z-ideal. Let $C_c(X) = \{f \in C(X) : |f(X)| \leq \aleph_0\}$ and $C^F(X) = \{f \in C(X) : |f(X)| < \infty\}$, see [6] and [7]. The socle of $C(X)$ (i.e., $C_F(X)$) which is in fact a direct sum of minimal ideals of $C(X)$ is characterized topologically in [10, Proposition 3.3], and it turns out that $C_F(X) = \{f \in C(X) : |X\backslash Z(f)| < \infty\}$ is a useful object in the context of $C(X)$, see [10], [1], [5], [2], and [3]. This motivates us to investigate the locally socle of $C(X)$. We define $LC_F(X) = \left\{f \in C(X) : \overline{S_f} = X\right\}$, where S_f is the union of all open subsets U in X such that $|U\backslash Z(f)| < \infty$, $LC_F(X)$ is called the locally socle of $C(X)$ and it is a z-ideal of $C(X)$ containing $C_F(X)$. We characterize spaces X for which the equality in the relation $C_F(X) \subseteq LC_F(X) \subseteq C(X)$ holds. In fact, we show that X is an almost discrete space if and only if $LC_F(X) = C(X)$. We note that if X is an infinite space, then $C_F(X) \subsetneq C(X)$. We also observe that $|I(X)| < \infty$ if and only if $C_F(X) = LC_F(X)$. Moreover, it is shown that if $|I(X)| < \infty$, then $LC_F(X)$ is never essential in any subring of $C(X)$, while $LC_F(X)$ is an intersection of essential ideals of $C(X)$. We determine the conditions such that $LC_F(X)$ is not prime in any subrings of $C(X)$ which contains the idempotents of X. We investigate the primness of $LC_F(X)$ in some subrings of $C(X)$. All

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