On a new notion of injectivity of Banach modules

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Abstract

In this paper, we introduce a new homological properties of Banach modules. It is shown that for a locally compact group G, the dual space of all bounded left uniformly continuous functions $LUC(G)'$ is 0-injective in the category of left Banach $M(G)$-modules.

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1 preliminaries

Let A be a Banach algebra and $\Delta(A)$ denote the character space of A, i.e., the space of all non-zero homomorphisms from A onto $\mathbb{C}$. We denote by **A-mod** and **mod-A** the category of all Banach left A-modules and all Banach right A-modules respectively. In the case that A has an identity we denote by **A-unmod** the category of all Banach left unital modules. For $E, F \in$ **A-mod**, let ${}_AB(E, F)$ be the space of all bounded linear left A-module morphisms from E into F.

Let $E, F \in$ **A-mod**. Suppose that $Z^1(A \times E, F)$ denotes the Banach space of all continuous bilinear maps $B : A \times E \longrightarrow F$ satisfying

$$a \cdot B(b, \xi) - B(ab, \xi) + B(a, b \cdot \xi) = 0 \quad (a, b \in A, \xi \in E).$$

Define $\delta_0 : B(E, F) \longrightarrow Z^1(A \times E, F)$ by $(\delta_0 T)(a, \xi) = a \cdot T(\xi) - T(a \cdot \xi)$ for all $a \in A$ and $\xi \in E$. Then we have

$$\mathrm{Ext}^1_A(E, F) = Z^1(A \times E, F)/\mathrm{Im}\delta_0.$$

By [6, Proposition VII.3.19], we know that $\mathrm{Ext}^1_A(E, F)$ is topologically isomorphic to $H^1(A, B(E, F))$ where $B(E, F)$ is a Banach A-bimodule with the following module actions:

$$(a \cdot T)(\xi) = a \cdot T(\xi), \quad (T \cdot a)(\xi) = T(a \cdot \xi) \quad (a \in A, \xi \in E, T \in B(E, F)).$$

To see further details about $\mathrm{Ext}^1_A(E, F)$; see [7].

Definition 1.1. Let A be a Banach algebra and $J \in$ **A-mod**. We say that J is injective if for each $F, E \in$ **A-mod** and admissible monomorphism $T : F \to E$ the induced map $T_J : {}_AB(E, J) \to {}_AB(F, J)$ defined by $T_J(R) = R \circ T$ is onto.

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