A classification of Ramanujan complements of unitary Cayley graphs

Reza Safakish*
University of Buali sina

Shahram mehry
University of Buali sina

Abstract

The unitary Cayley graph on n vertices, X_n, has vertex set $\mathbb{Z}_n$, and two vertices a and b are connected by an edge if and only if they differ by a multiplicative unit modulo n, i.e. $\gcd(ab, n) = 1$. A k-regular graph X is Ramanujan if and only if $\lambda(X) \leq 2\sqrt{k-1}$ where $\lambda(X)$ is the second largest absolute value of the eigenvalues of the adjacency matrix of X. We obtain a complete characterization of the cases in which the complements of unitary Cayley graph $\bar{X}_n$ is a Ramanujan graph.

Keywords: Graph, Cayley Graph, Ramanujan Graph

1 Introduction

We define the Cayley graph $X = Cay(G, S)$ to be the graph whose vertex set is G, and in which two vertices v and u in G are connected by an edge if and only if vu^{-1} is in S.

The unitary Cayley graph on n vertices, X_n, is defined to be the undirected graph whose vertex set is $\mathbb{Z}_n$, and in which two vertices a and b are connected by an edge if and only if $\gcd(a-b, n) = 1$. This can also be stated as $X_n = Cay(\mathbb{Z}_n, \mathbb{U}_n)$, where $\mathbb{Z}_n$ is the additive group of integers modulo n and $\mathbb{U}_n = \mathbb{Z}_n^*$ is the set of multiplicative units modulo n. X_n is a simple, $\varphi(n)$-regular graph, where φ is the Euler totient function. Here $\varphi(n)$ is defined by $\varphi(1) = 1$, and for an integer $n > 1$ with distinct prime power factorization $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ for distinct primes $p_1, \ldots, p_k$ and nonnegative integers $\alpha_1, \ldots, \alpha_k$, with $k > 0$, $\varphi(n) = p_1^{\alpha_1-1} p_2^{\alpha_1-1} \cdots p_k^{\alpha_k-1}(p_1-1)(p_2-1)\cdots(p_k-1)$. When discussing X_n, we always assume $n > 3$

Lemma 1.1. *The eigenvalues of any adjacency matrix of X_n are*

$$\lambda_m(n) = \mu\left(\frac{n}{(n,m)}\right)\frac{\varphi(n)}{\varphi\left(\frac{n}{(n,m)}\right)} \tag{1}$$

Proof. see [3, Klotz, W. and Sander, T. (2007)] □

When $\frac{n}{(n,m)}$ is square-free,

$$|\lambda_m(n)| = \frac{\varphi(n)}{\varphi\left(\frac{n}{(n,m)}\right)} \tag{2}$$

*Speaker