Derivations on the algebra of operators in Hilbert modules over locally C*-algebras

Khadijeh Karimi*
Shahrood University

Kamran Sharifi
Shahrood University

Abstract

Let E be a Hilbert module over a locally-C*-algebra $\mathcal{A}$ and $\mathcal{L}_{\mathcal{A}}(E)$ be the algebra of all adjointable $\mathcal{A}$-module operators on E. We show that if $\mathcal{A}$ is a unital commutative locally-C*-algebra and $b(E)$, the set of all bounded elements of E, is a full Hilbert $b(A)$-module then every derivation on $\mathcal{L}_{\mathcal{A}}(E)$ is inner. If $\mathcal{A}$ be a commutative σ-C*-algebra with a countable approximate unit and E is full, then every derivation on $\mathcal{L}_{\mathcal{A}}(E)$ is a weakly approximately inner derivation. Moreover, the innerness of derivations on compact operators implies the innerness of derivations on $\mathcal{L}_{\mathcal{A}}(E)$.

Keywords: Hilbert modules, Locally C*-algebras, Derivations
Mathematics Subject Classification [2010]: 46L08, 46L05, 46L57

1 Introduction

Recall that a derivation of an algebra $\mathcal{A}$ is a linear mapping Δ from $\mathcal{A}$ into itself, such that $\Delta(ab) = \Delta(a)b + a\Delta(b)$ for all $a, b \in \mathcal{A}$. We say that Δ is inner if there exists $x \in \mathcal{A}$ such that $\Delta(a) = [a, x] = ax - xa$ for every $a \in \mathcal{A}$. One of the interesting problem in the theory of derivations is to identify those algebras on which all the derivations are inner, i.e. the first cohomology group is trivial. The first result of this problem is probably due to Kaplansky [6] who proved that every derivation of a type I W*-algebra is inner. In 1966, Sakai [8] extended the result of Kaplansky and proved that every derivation of a W*-algebra is inner. Finally Kadison [5] proved the innerness of derivation on von Neumann algebras.

A *locally C*-algebra* is a complete Hausdorff complex topological $*$-algebra $\mathcal{A}$ whose topology is determined by its continuous C*-seminorms in the sense that the net $\{a_i\}_{i\in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for every continuous C*-seminorm p on $\mathcal{A}$. A *σ-C*-algebra* is a locally C*-algebra whose topology is determined by a countable family of C*-seminorms. These algebras were first introduced by Inoue [3] as a generalization of C*-algebras and appear in the study of certain aspects of C*-algebras such as tangent algebras of C*-algebras, domain of closed $*$-derivations on C*-algebras, multipliers of Pedersen's ideal, noncommutative analogues of classical Lie groups, and K-theory. Let $\mathcal{S}(\mathcal{A})$ be the set of all continuous C*-seminorms on $\mathcal{A}$. For $p \in \mathcal{S}(\mathcal{A})$, $\mathcal{A}_p = \mathcal{A}/N_p$, where $N_p = \{a \in \mathcal{A} : p(a) = 0\}$ is a C*-algebra in the norm induced by

*Speaker