On some means inequalities in matrix spases

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Abstract

In this paper, we state some recent results on non-commutative version of refinements and reverses of ν-weighted arthimetic-geometric-harmonic mean inequality, which is a fundamental relation between two nonnegative real numbers, in the frame work of matrices.

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1 Introduction

The well-known Young inequality, states that if a, b are two positive numbers and $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

and equality holds if and only if $a = b$. Equivalently, for distinct positive numbers a, b and $0 < \nu < 1$, we have

$$a^\nu b^{1-\nu} < \nu a + (1-\nu)b.$$

By defining weighted arithmetic and geometric means as $A_\nu(a, b) = \nu a + (1-\nu)b$ and $G_\nu(a, b) = a^\nu b^{1-\nu}$, respectively, the Young inequality can be written as $G_\nu(a, b) < A_\nu(a, b)$, which is known as the arithmetic-geometric mean inequality. A similar inequality, known as geometric-harmonic mean inequality, states that $H_\nu(a, b) < G_{nu}(a, b)$ where $H_\nu(a, b) = (\nu a^{-1} + (1-\nu)b^{-1})^{-1}$ is the harmonic mean of a, b.

One can consider these inequalities on the complex matrix space.

Definition 1.1. For two positive definite matrices A, B, we define

- arithmetic mean of A, B:

$$A \nabla_\nu B = \nu A + (1-\nu)B,$$

- geometric mean of A, B:

$$A \sharp_\nu B = A^{1/2}(A^{-1/2} B A^{-1/2})^{1-\nu} A^{1/2},$$

- harmonic mean of A, B:

$$A !_\nu B = (\nu A^{-1} + (1-\nu)B^{-1})^{-1}.$$