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Phase portraits of separable Hamiltonian systems☆

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ABSTRACT

We study a generalization of potential Hamiltonian systems ($H(x, y) = y^2 + F(x)$) with one degree of freedom; namely, those with Hamiltonian functions of type $H(x, y) = F(x) + G(y)$, which will be denoted by X_H. We present an algorithm to obtain the phase portrait (including the behaviour at infinity) of X_H when F and G are arbitrary polynomials. Indeed, from the graphs of the one-variable functions F and G, we are able to give the full description on the Poincaré disk, therefore extending the well-known method to obtain the phase portrait of potential systems in the finite plane. The fact that the phase portraits can be fully described in terms of the two one-variable real functions F and G allows, as well, a complete study of the bifurcation diagrams in complete families of vector fields. The algorithm can be applied to study separable Hamiltonian systems with one degree of freedom, which include a vast amount of examples in physical applications.

1. Introduction

Hamiltonian systems are ubiquitous in mathematical physics, specially in mechanics, but also in control engineering, biology and other fields. The goals in the study of Hamiltonian systems are diverse, according to the dimension (number of degrees of freedom) and the complexity of the Hamiltonian function H. In this paper, we deal with a relatively simple family of Hamiltonian systems (low dimensional and polynomial) but, in compensation, we are able to give an algorithm to plot the phase portrait including the behaviour at infinity, that is, to provide the full qualitative description of the associated dynamics. In particular, the algorithm allows the study of families of vector fields and their bifurcations in a rather simple way. Our object is the Hamiltonian system with one degree of freedom and separated variables, and the energy function can be written as:

$$H(x, y) = F(x) + G(y), \tag{1}$$

where x represents the phase, usually called q in the literature, and y the momentum, usually called p.

The main reason for the choice of this family is the balance between applicability and feasibility: on one hand, there is a vast literature on separable Hamiltonian systems and, on the other hand, there are no general algorithms to systematically study them except for the case $G(y) = y^2/2$. Thus, the results presented here can be potentially applied to relevant physical systems (see next paragraph), by means of a systematic procedure to obtain the full qualitative description of the orbits.

One can find examples of Hamiltonian systems of type (1) in classical textbooks, see for instance [1] and [2, Chap. II], but also more recent and specific examples can be found, for instance, in relativistic potentials or fluid kinetics. In the context of relativistic mechanics, [3], for instance, study constant period oscillators in the family $H(p, q) = \sqrt{p^2 c^2 + m^2 c^4} - m c^2 +$

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