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On the Dirichlet problem for the n, α-Laplacian with the nonlinearity in the critical growth range

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ABSTRACT

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. Applying the Mountain Pass Theorem we prove the existence of a non-trivial weak solution to the Dirichlet problem

$$-\operatorname{div}\left(\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = f(x,u) \quad \text{in } \Omega,$$

where u is in the Orlicz–Sobolev space $W_0^1 L^\Phi(\Omega)$ with a Young function of the type $\Phi(t) \approx t^n \log^\alpha(t)$, $\alpha < n-1$, and $|f(x,t)| \approx \exp(\beta |t|^{\frac{n}{n-1-\alpha}})$, $\beta > 0$.

1. Introduction

Throughout the paper, Ω is supposed to be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and ω_{n-1} denotes the $(n-1)$-dimensional measure of the surface of the unit sphere.

It is a well-known problem to find solutions to the Laplace equation

$$u \in W_0^{1,2}(\Omega) \quad \text{and} \quad -\Delta u = f(x,u) \quad \text{in } \Omega. \tag{1.1}$$

When $n \geq 3$ and f satisfies $\lim_{t\to\infty} \frac{f(x,t)}{t^q} = 0$ uniformly on Ω with $q < \frac{n+2}{n-2}$, then there are many results using the compact embedding of the space $W_0^{1,2}(\Omega)$ into $L^r(\Omega)$ with $r \in [1, \frac{2n}{n-2})$ (see a review article by Lions [1] and the references given therein). Problem (1.1) under condition $\lim_{t\to\infty} f(x,t)t^{-\frac{n+2}{n-2}} = 0$ becomes much more difficult thanks to the fact that the embedding of the Sobolev space $W_0^{1,2}(\Omega)$ into $L^{\frac{2n}{n-2}}(\Omega)$ is no longer compact. This difficulty was overcome by Brezis and Nirenberg [2] using the Mountain Pass Theorem by Ambrosetti and Rabinowitz [3].

When $n \geq 2$, we not only have the Sobolev embedding of $W_0^{1,n}(\Omega)$ into $L^r(\Omega)$ for any $r \in [0,\infty)$ but also have the Trudinger embedding [4] into the Orlicz space $\exp L^{\frac{n}{n-1}}(\Omega)$. In particular, there is the so-called Moser–Trudinger inequality by Moser [5]

$$\sup_{\|u\|_{W_0^{1,n}(\Omega)} \leq 1} \int_\Omega \exp(K|u|^{\frac{n}{n-1}}) \leq C(n, \mathcal{L}_n(\Omega)) \quad \text{if and only if} \quad K \leq n\omega_{n-1}^{\frac{1}{n-1}}.$$

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