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On some problems on smooth approximation and smooth extension of Lipschitz functions on Banach–Finsler manifolds

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ABSTRACT

Let us consider a Riemannian manifold M (either separable or non-separable). We prove that, for every $\varepsilon > 0$, every Lipschitz function $f : M \to \mathbb{R}$ can be uniformly approximated by a Lipschitz, C^1-smooth function g with $\mathrm{Lip}(g) \leq \mathrm{Lip}(f) + \varepsilon$. As a consequence, every Riemannian manifold is uniformly bumpable. These results extend to the non-separable setting those given in [1] for separable Riemannian manifolds. The results are presented in the context of C^ℓ Finsler manifolds modeled on Banach spaces. Sufficient conditions are given on the Finsler manifold M (and the Banach space X where M is modeled), so that every Lipschitz function $f : M \to \mathbb{R}$ can be uniformly approximated by a Lipschitz, C^k-smooth function g with $\mathrm{Lip}(g) \leq C\mathrm{Lip}(f)$ (for some C depending only on X). Some applications of these results are also given as well as a characterization, on the separable case, of the class of C^ℓ Finsler manifolds satisfying the above property of approximation. Finally, we give sufficient conditions on the C^1 Finsler manifold M and X, to ensure the existence of Lipschitz and C^1-smooth extensions of every real-valued function f defined on a submanifold N of M provided f is C^1-smooth on N and Lipschitz with the metric induced by M.

1. Introduction

In this work we address the problem as to whether every Lipschitz function $f : M \to \mathbb{R}$ defined on a *non-separable* Riemannian manifold can be uniformly approximated by a Lipschitz, C^∞-smooth function $g : M \to \mathbb{R}$. The study of this problem was motivated by the work in [1], where this result of approximation is stated for *separable* Riemannian manifolds. The question whether this result holds for every Riemannian manifold is posed in [2,1,3]. A positive answer to this question provides nice applications such as: (i) the uniformly bumpable character of every Riemannian manifold, (ii) Deville–Godefroy–Zizler smooth variational principle holds for every complete Riemannian manifold [2] and (iii) the infinite-dimensional version of the Myers–Nakai theorem given in [3] holds for every complete Riemannian manifold (either separable or non-separable) as well.

The problem of the uniform approximation of real-valued and Lipschitz functions defined on a Banach space by Lipschitz and C^1-smooth functions has been largely studied. J.M. Lasry and P.L. Lions used sup-inf convolution techniques to answer positively to this problem on every Hilbert space [4]. R. Fry introduced "sup-partitions of unity", a key tool to answer positively to this problem for the case of bounded and real-valued Lipschitz functions defined on Banach spaces with separable dual [5]. Later on, D. Azagra et al. extended this result in [6]. In particular, they obtained C^k smoothness of the Lipschitz approximating functions whenever X is separable and admits a Lipschitz C^k-smooth bump function. Recently, it

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