Existence of nonnegative solutions for quasilinear elliptic equations with indefinite critical nonlinearities

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ABSTRACT

We study the quasilinear elliptic equation

$$-\text{div}(\phi(|\nabla u|)\nabla u) = a(x)f(u) + b(x)g(u) \quad \text{in } \Omega$$

with Dirichlet boundary condition $u = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbf{R}^N$, $a(x)$, $b(x)$ are sign-changing continuous functions, and $g(u)$ has critical growth at infinity with respect to the principal part ϕ. A nonnegative, nontrivial solution is given under appropriate growth conditions on $f(u)$, $g(u)$ at 0 and ∞.

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1. Introduction

Let us consider a quasilinear elliptic boundary value problem in the form

$$\begin{cases} -\text{div}(\phi(|\nabla u|)\nabla u) = a(x)f(u) + b(x)g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^N$, $N \geq 2$, $a(x)$, $b(x)$ are continuous functions in Ω, which may change sign, $f(u)$ and $g(u)$ are nonnegative continuous functions. The function $\phi(t)$ appearing in the principal part is assumed to grow as t^{p-2} asymptotically as $t \to \infty$ and $g(t)$ has critical growth, that is, $g(t) = O(t^{p^*})$ as $t \to \infty$ with $p^* = Np/(N-p)$.

The type of Eq. (1.1) with $\phi(t) = t^{p-2}$ is well known as the p-Laplace equation involving a critical Sobolev exponent and has been investigated extensively. The existence and multiplicity of positive solutions for the p-Laplace equation in a bounded domain with a positive definite lower term were discussed by many authors, e.g., [1–4]. Further, de Figueiredo–Gossez–Ubilla [5,6] studied the p-Laplace equation with an indefinite lower term. In addition, subcritical problems of the p&q-Laplacian type equation

$$-\text{div}(|\nabla u|^{p-2}\nabla u) - \text{div}(|\nabla u|^{q-2}\nabla u) + a(x)|u|^{p-2} + b(x)|u|^{q-2} = f(x, u) \tag{1.2}$$

were studied by several authors [7–9].

The problems on the whole space $\mathbf{R}^N$ have also been studied in, for example, [10–12]. Among them, Silva and Soares [12] considered (1.1) for the p-Laplace equation on $\mathbf{R}^N$ and discussed the existence of nonnegative nontrivial solutions. Following

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