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On best constants in Hardy inequalities with a remainder term

Salvatore Cuomo [1], Adamaria Perrotta *

Dipartimento di Matematica ed applicazioni "R.Caccioppoli", Complesso Universitario M. S. Angelo, Via Cinthia, 80126 Napoli, Italy

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ABSTRACT

Let Ω be a bounded open set of $\mathbb{R}^N$ containing the origin. We compute the best value of the constant $C(\alpha, |\Omega|)$ in

$$\int_\Omega |\nabla u|^2\, dx - \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx \geq C(\alpha, |\Omega|)\, \|u\|^2_{L\left(\frac{2N}{N-\alpha}, 2\right)},$$

with $\alpha < 2$ and $u \in H^1_0(\Omega)$. Then we get the optimal value of $C(|\Omega|)$ in

$$\int_\Omega |\nabla u|^3\, dx - \left(\frac{N-3}{3}\right)^3 \int_\Omega \frac{u^3}{|x|^3} dx \geq C(|\Omega|)\, \|u\|^3_{L^3},$$

where $u \in W^{1,3}_0(\Omega)$.

1. Introduction

Let $N > 2$ and let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain containing the origin.

The well known Hardy–Sobolev inequality (see [1,2]) reads

$$\int_\Omega |\nabla u|^p\, dx \geq \left(\frac{N-p}{p}\right)^p \int_\Omega \frac{u^p}{|x|^p} dx, \quad \forall u \in W^{1,p}_0(\Omega). \tag{1.1}$$

This inequality and its various improvements are used in many contexts, as in the study of stability of solutions of semilinear elliptic and parabolic equations (see [3–5]), or in the analysis of the asymptotic behavior of the heat equation with singular potentials (see [6]).

The constant $C_{N,p} = \left(\frac{N-p}{p}\right)^p$ is the best one and there is no function $u \in W^{1,p}_0(\Omega)$ for which it is achieved. For this reason several authors have improved inequality (1.1) by adding a nonnegative correction term.

In case $p = 2$ the first result is due to Brezis and Vazquez; in [3] they prove the so-called Hardy–Poincaré inequality

$$\int_\Omega |\nabla u|^2\, dx - \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx \geq \Lambda_2 \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \|u\|^2_{L^2}, \quad \forall u \in H^1_0(\Omega). \tag{1.2}$$

Here Λ_2 denotes the first eigenvalue of the Laplace operator in the two dimensional unit disk, and ω_N and $|\Omega|$ are respectively the N-dimensional Lebesgue measure of the unit ball $B_1(0) \subseteq \mathbb{R}^N$ and of the set Ω. The value $\Lambda_2 \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}}$ is optimal in the ball but it is not achieved in $H^1_0(\Omega)$.

* Corresponding author. Tel.: +39 081675703; fax: +39 081675636.
E-mail addresses: salvatore.cuomo@unina.it (S. Cuomo), adamaria.perrotta@unina.it (A. Perrotta).

1 Tel.: +39 081675624; fax: +39 081675636.

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