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Existence and regularity of nonnegative solution of a singular quasi-linear anisotropic elliptic boundary value problem with gradient terms

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ABSTRACT

In this paper, we consider the singular quasi-linear anisotropic elliptic boundary value problem

$$\begin{cases} f_1(u)u_{xx} + u_{yy} + g(u)|\nabla u|^q + f(u) = 0, & (x, y) \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{P}$$

where Ω is a smooth, bounded domain in R^2; $0 < q < 2$; $f_1(0) = 0, f_1(t) > 0\ (t \neq 0), f_1$ is a smooth function in $(-\infty, +\infty)$ and is a non-decreasing function in $(0, +\infty)$; $g(t) \geq 0$, g is a smooth function in $(-\infty, 0) \cup (0, +\infty)$ and is a non-increasing function in $(0, +\infty)$; $f(t) > 0, f$ is a smooth function in $(-\infty, 0) \cup (0, +\infty)$ and is a strictly decreasing function in $(0, +\infty)$. Clearly, this is a boundary degenerate elliptic problem if $f_1(0) = 0$. We show that the solution of the Dirichlet boundary value problem (P) is smooth in the interior and continuous or Lipschitz continuous up to the degenerate boundary and give the conditions for which gradients of solutions are bounded or unbounded. We believe that these results on regularity of the solution should be very useful.

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1. Introduction

Consider a singular quasi-linear anisotropic elliptic boundary value problem

$$\sum_{i=1}^{N-1} f_i(u)u_{x_ix_i} + u_{x_nx_n} + g(u)|\nabla u|^q + f(u) = 0, \quad \text{in } \Omega, \tag{1.1}$$

$$u|_{\partial\Omega} = 0, \tag{1.2}$$

where Ω is a bounded domain in R^N, $0 < q \leq 2$. This problem arises in certain applications in fluid mechanics and pseudoplastic flow (see [1–4]). The N-dimensional problem (1.1)–(1.2) has been studied in [4–15] for some special forms.

In fact, the problem (1.1)–(1.2) is a special case of a quasi-linear anisotropic elliptic boundary value problem which arises naturally in studying self-similar solutions of hyperbolic conservation laws in two dimensions (see [6]). Some results for anisotropic elliptic boundary value problems can be found in [5–12] with $g = 0$.

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