Nonlinear Analysis 74 (2011) 5525–5543

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Comparisons of relative BV-capacities and Sobolev capacity in metric spaces

Heikki Hakkarainen [a,*], Nageswari Shanmugalingam [b]

[a] Department of Mathematical Sciences, P.O. Box 3000, FI-90014, University of Oulu, Finland

[b] Department of Mathematical Sciences, P.O. Box 210025, University of Cincinnati, Cincinnati, OH 45221–0025, USA

ARTICLE INFO

Article history:
Received 29 March 2011
Accepted 12 May 2011
Communicated by Enzo Mitidieri

MSC:
28A12
26A45
30L99

Keywords:
Capacity
Functions of bounded variation
Sobolev space
Analysis on metric measure spaces

ABSTRACT

We study relations between the variational Sobolev 1-capacity and versions of variational BV-capacity in a complete metric space equipped with a doubling measure and supporting a weak (1, 1)-Poincaré inequality. We prove the equality of 1-modulus and the continuous 1-capacity, extending the known results for $1 < p < \infty$ to also cover the more geometric case $p = 1$. Then we give alternative definitions for variational BV-capacities and obtain equivalence results between them. Finally we study relations between total 1-capacity and versions of BV-capacity.

1. Introduction

In this article we study connections between different 1-capacities and BV-capacities in the setting of metric measure spaces. We obtain characterizations for the 1-capacity of a condenser, extending results known for $p > 1$ to also cover the more geometric case $p = 1$. Furthermore, we study how the different versions of BV-capacity, corresponding to various pointwise requirements that the capacity test functions need to fulfill, relate to each other.

One difficulty that occurs when working with minimization problems on the space $W^{1,1}(\mathbb{R}^n)$ is the lack of reflexivity. Indeed, the methods used to develop the theory of p-capacity are closely related to those used in certain variational minimization problems; in such problems reflexivity or the weak compactness property of the function space $W^{1,p}(\mathbb{R}^n)$ when $p > 1$ usually plays an important role. One possible way to deal with this lack of reflexivity is to consider the space $\mathrm{BV}(\mathbb{R}^n)$, that is, the class of functions of bounded variation. This wider class of functions provides tools, such as lower semicontinuity of the total variation measure, that can be used to overcome the problems caused by the lack of reflexivity in the arguments.

This approach was originally used to study variational 1-capacity in the Euclidean case in [1–3]. The article [4] showed that a similar approach could also be used to study variational 1-capacity in the setting of metric measure spaces. In [4] the main tool in obtaining a connection between the 1-capacity and BV-capacity was the metric space version of Gustin's boxing inequality, see [5]. Since then, this strategy has been used in [6] to study a version of BV-capacity and Sobolev 1-capacity in the setting of metric measure spaces. Since the case $p = 1$ corresponds to geometric objects in the metric measure space

* Corresponding author. Tel.: +358 443502482.
E-mail addresses: heikki.hakkarainen@oulu.fi (H. Hakkarainen), nages@math.uc.edu (N. Shanmugalingam).

0362-546X/$ – see front matter
doi:10.1016/j.na.2011.05.036