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Three positive solutions for a semilinear elliptic equation in $\mathbb{R}^N$ involving sign-changing weight

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ABSTRACT

In this paper, we study the decomposition of the Nehari manifold via the combination of concave and convex nonlinearities. Furthermore, we use this result and the Lusternik–Schnirelmann category to prove that a semilinear elliptic equation in $\mathbb{R}^N$ involving sign-changing weight function has at least three positive solutions.

1. Introduction

In this paper, we prove the multiplicity results of positive solutions of the following semilinear elliptic problem:

$$\begin{cases} -\Delta u + u = f_\lambda(x)|u|^{q-2}u + |u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{E_{f_λ}}$$

where $N \geq 3, 1 < q < \min\left\{2, \left(1+\frac{4}{N}\right)\right\}, 2 < p < \frac{2N}{N-2}, \lambda > 0, f_\lambda = \lambda f_+(x) - f_-(x)$ and the functions $f_\pm$ satisfy the following conditions:

(D1) there exists a bounded smooth domain $\Theta \subset \mathbb{R}^N$ such that for all $x \in \Theta, f_-(x) \geq 0,\ f_+(x) = 0$ and for all $x \in \mathbb{R}^N \setminus \overline{\Theta}, f_-(x) = 0, f_+(x) \geq 0$;

(D2) $f_+(x) \in [L^{q^*}(\mathbb{R}^N) \cap L^{\widetilde{q}}(\mathbb{R}^N)] \setminus \{0\}$, where $q^* = \frac{2}{2-q}$ and $\widetilde{q} = \frac{2\beta}{2-(q-1)\beta}$ $\left(\beta = \frac{N}{2}, \text{if } N = 3 \text{ and } \frac{N}{2} < \beta < \frac{2}{q-1}, \text{if } N \geq 4\right)$;

(D3) $f_-(x) \in L^{\overline{q}}(\mathbb{R}^N) \setminus \{0\}$, where $\overline{q} = \max\{q^*, \widetilde{q}\}$.

Associated with Eq. (E_{f_λ}), we consider the energy functional J_{f_λ} in $H^1(\mathbb{R}^N)$

$$J_{f_\lambda}(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{q}\int_{\mathbb{R}^N} f_\lambda(x)|u|^q dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx$$

where $\|u\|_{H^1} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx\right)^{1/2}$ is the standard norm in $H^1(\mathbb{R}^N)$. It is well known that the solutions of Eq. (E_{f_λ}) are the critical points of the energy functional J_{f_λ} in $H^1\left(\mathbb{R}^N\right)$ (see [1]).

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