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Fractional differential equations and Lyapunov functionals

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ABSTRACT

We consider a scalar fractional differential equation, write it as an integral equation, and construct several Lyapunov functionals yielding qualitative results about the solution. It turns out that the kernel is convex with a singularity and it is also completely monotone, as is the resolvent kernel. While the kernel is not integrable, the resolvent kernel is positive and integrable with an integral value of one. These kernels give rise to essentially different types of Lyapunov functionals. It is to be stressed that the Lyapunov functionals are explicitly given in terms of known functions and they are differentiated using Leibniz's rule. The results are readily accessible to anyone with a background of elementary calculus.

1. Introduction

We study a fractional differential equation of Caputo type

$$^{c}D^{q}x = f(t, x(t)), \quad 0 < q < 1, \tag{1}$$

with $f : [0, \infty) \times \Re \to \Re$ being continuous. Because it is of Caputo type it is inverted with simple initial conditions, just as an ordinary differential equation [1, p. 12], and written as

$$x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds \tag{2}$$

where Γ is the gamma function. We refer the reader to Lakshmikantham et al. [1, p. 54] or to Chapter 6 of Diethelm [2, pp. 78, 86,103] for proofs of the inversion. It is to be emphasized that (1) and (2) are not equivalent for the Riemann–Liouville derivative denoted by D^q instead of $^{c}D^q$ which is used in some of the papers listed below. In fact, Caputo introduced his derivative to avoid the initial conditions imposed by the Riemann–Liouville derivative which were difficult to reconcile with many real-world problems. Having made the point that (1) is of Caputo type, we now point out that there are initial conditions in the Riemann–Liouville problem which can sometimes be incorporated under the integral in (2) and $x(0)$ can be deleted. That will not be discussed further here.

Before the reader's patience wanes, let us quickly sketch what we accomplish here. The equation on which we focus is

$$x(t) = x(0) + F(t) - \frac{k}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x(s)) ds \tag{3}$$

where $x(0)$ is sometimes 0 and where either

$$xg(t, x) \geq 0 \quad \text{or} \quad g(t, x) = x + G(t, x) \quad \text{with } |G(t, x)| \leq \phi(t)|x| \tag{4}$$

and ϕ is small in several different ways.

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