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Boundary value problems for mixed type equations and applications

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ABSTRACT

In this paper we outline a general method for finding well-posed boundary value problems for linear equations of mixed elliptic and hyperbolic type, which extends previous techniques of Berezanskii, Didenko, and Friedrichs. This method is then used to study a particular class of fully nonlinear mixed type equations which arise in applications to differential geometry.

1. Introduction

An old classical problem from differential geometry asks, when can one realize a two-dimensional Riemannian manifold, locally, in a three-dimensional Euclidean space? In other words, when can one "see" an abstract surface, at least locally? As it turns out, this question is equivalent to finding local solutions $z(x, y)$ to a Monge–Ampère type equation, referred to as the Darboux equation:

$$\det \nabla_{ij} z = K(\det h)(1 - |\nabla_h z|^2). \tag{1.1}$$

Here h is the given Riemannian metric, ∇_{ij} are second covariant derivatives, and K is the Gaussian curvature of h. Another related problem is that of locally prescribing the Gaussian curvature of surfaces in three-dimensional Euclidean space. More precisely, given a function $K(x, y)$ defined in a neighborhood of the origin, does there exist a graph $z = z(x, y)$ having Gaussian curvature K? Note that every surface may be expressed locally as a graph. This problem is also equivalent to the local solvability of a Monge–Ampère equation, namely

$$\det \partial_{ij} z = K(1 + |\nabla z|^2)^2, \tag{1.2}$$

where ∂_{ij} are second partial derivatives. In both Eqs. (1.1) and (1.2), the sign of the Gaussian curvature completely determines the type of the equation. When K is positive the equation is elliptic, and when K is negative the equation is hyperbolic. Thus classical results may be used to analyze these problems in these two situations. However when K changes sign, the equation is of mixed type, and is very difficult to study. Nevertheless, it can be shown [1] that by a suitable application of a Nash–Moser iteration, these two problems reduce to the study of a linear equation having a particular form described below. More precisely, in order to successfully apply the Nash–Moser iteration, one must find a well-posed boundary value problem for the associated linearized equation, in a fixed domain about the origin, and establish certain a priori estimates. In previous work by Han, Hong, Lin, as well as the author, this has been accomplished in the case for which the Gaussian curvature changes sign to finite order along a single smooth curve (see [2–6]), and also in the case for which the Gaussian curvature vanishes to finite order and has a zero set consisting of two transversely intersecting curves (see [1,7]). Our goal

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