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Multiple positive solutions for semilinear elliptic equations with critical weighted Hardy–Sobolev exponents ☆

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ABSTRACT

The existence and multiplicity of positive solutions are obtained for a class of semilinear elliptic equations with critical weighted Hardy–Sobolev exponents and the concave–convex nonlinearity by variational methods and some analysis techniques.

1. Introduction and main results

Consider the following semilinear elliptic problem

$$\begin{cases} -\text{div}(|x|^{-2a}\nabla u) - \mu \dfrac{u}{|x|^{2(1+a)}} = \dfrac{|u|^{p-2}}{|x|^{bp}}u + \lambda f(x)u + \nu g(x,u), & x \in \Omega \setminus \{0\}, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $R^N (N \geq 3), 0 \in \Omega, 0 \leq a < \sqrt{\overline{\mu}}, 0 \leq \mu < (\sqrt{\overline{\mu}} - a)^2$ with $\overline{\mu} \triangleq \frac{(N-2)^2}{4}, a \leq b < a+1$, λ and ν are positive parameters, $p = p(a,b) \triangleq \frac{2N}{N-2(1+a-b)}$ is the Hardy–Sobolev critical exponent. Note that $p(a,a) = \frac{2N}{N-2} = 2^*$ is the Sobolev critical exponent. $f(x)$ is a positive measurable function, $g \in C(\overline{\Omega} \times R, R)$, $G(x,t)$ is the primitive function of $g(x,t)$ defined by $G(x,t) = \int_0^t g(x,s)\mathrm{d}s$ for $x \in \Omega, t \in R$.

Problem (1) is of the form

$$-\text{div}(A(x)\nabla u) = f(x,u), \tag{2}$$

where $A(x)$ is a nonnegative function which may have zeros at some points and may be unbounded. We also remark that equation (2) comes from the consideration of standing waves in the anisotropic Schrödinger equation.

To solve the problem variationally, we depend much on the following famous Caffarelli–Kohn–Nirenberg inequalities in [1],

$$\left(\int_{R^N} |x|^{-bp}|u|^p \mathrm{d}x\right)^{\frac{2}{p}} \leq C_{a,b} \int_{R^N} |x|^{-2a}|\nabla u|^2 \mathrm{d}x, \quad \text{for all } u \in C_0^\infty(R^N). \tag{3}$$

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