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Second-order subdifferentials and convexity of real-valued functions ☆

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ABSTRACT

In this paper we show that the positive semi-definiteness (PSD) of the Fréchet and/or Mordukhovich second-order subdifferentials can recognize the convexity of C^1 functions. However, the PSD is insufficient for ensuring the convexity of a locally Lipschitz function in general. A complete characterization of strong convexity via the second-order subdifferentials is also given.

1. Introduction

Convex functions appear and are useful in many areas of mathematics, including the calculus of variations, control theory, inequalities and functional equations, optimization, econometrics, and numerous applications in industry, business, medicine, and art. As far as we know, the second-order derivative test in calculus gives us a powerful tool in recognizing convexity of a function. For example, a C^2 function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for every $x \in \mathbb{R}^n$ the Hessian $\nabla^2\varphi(x)$ is a positive semi-definite matrix. Relaxing the assumption on the C^2 smoothness of the function under consideration, several authors have characterized the convexity by using various kinds of generalized second-order directional derivatives. We refer the reader to [1–7] for many interesting results obtained in this direction.

The *coderivative* of set-valued mappings, which was introduced by Mordukhovich [8], has been well recognized as a convenient tool to study many important issues in variational analysis and optimization. In [9], Poliquin and Rockafellar show that the positive definiteness of the *Mordukhovich* and/or *limiting second-order subdifferential* mapping $\partial^2\varphi(\bar{x}, 0) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ can characterize the *tilt stability* of a stationary point $\bar{x}$ of a function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ (provided that φ has some required properties). Furthermore, Levy et al. [10] have proved that the positive definiteness of a parametric limiting second-order subdifferential mapping can be used to study the *full stability* of local optimal points. Recently, as shown by [11], the convexity of a real-valued function can be characterized via the Fréchet and/or Mordukhovich second-order subdifferential mappings. More precisely, if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,1}$ function then the positive semi-definite property of one of the second-order subdifferentials, i.e.,

$$\langle z, u\rangle \geq 0 \quad \text{for all } u \in \mathbb{R}^n \text{ and } z \in \partial^2\varphi(x, y)(u) \text{ with } (x, y) \in \operatorname{gph} \partial\varphi, \tag{1.1}$$

is a complete characterization of the convexity of φ. A similar result for piecewise C^2 functions is also presented in [12].

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