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On the weak-star extensibility[☆]

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ABSTRACT

The paper encompasses a complete study of the weak-star extensibility of subspaces of Banach spaces introduced by B.S. Mordukhovich and B. Wang when studying the restrictive metric regularity in variational analysis. Many useful properties are presented; in particular, a new point of view and formulation of the property in the framework of Banach space theory is established. We also propose the concept of *weak-star extensible Banach spaces* that has interesting applications, and is fairly broad in that it contains many important classes of Banach spaces. Some further applications of this extensibility to the theory of linear operators between Banach spaces are also explored.

1. Introduction

From the Urysohn's extension theorem in point-set topology to the Hahn–Banach linear extension theorem in functional analysis, function extension has long been an important issue in analysis and its applications, and many of such results are cornerstones of the corresponding fields. Let X, Y be real Banach spaces and $L \subset X$ be a closed linear subspace. It is natural to ask the extensibility of a linear continuous mapping

$$T : L \to Y \tag{1.1}$$

to the whole space $\widetilde{T} : X \to Y$, which could be regarded as a generalization of the Hahn–Banach theorem in which case Y is the scalar field. Such an extension, unfortunately, is not always possible in general. Fixing the space Y, if such extensions exist for all Banach space X with its subspace L, and all continuous linear mapping T, then Y is called *injective*. Then $\mathbb{R}$ is injective by the Hahn–Banach Theorem. It is also well-known that ℓ^∞, the space of all bounded real sequences, is injective, while c_0, the space of all null real sequences, is not. In fact, the identity mapping $\mathbf{1}_{c_0} : c_0 \to c_0$ cannot be extended to $\ell^\infty \to c_0$; see more discussions in Section 3.

In the following, by w^* we mean the weak-star topology on the dual space X^* of X that is induced by $X \subset X^{**}$, and by a w^*-null sequence we mean a sequence that is w^*-convergent to 0. In [1], Mordukhovich and Wang introduced the *w^*-extensibility* of subspaces of a Banach space as below:

Definition 1.1. Let L be a closed subspace of a Banach space X. We say that L is w^*-extensible in X if every w^*-null sequence in L^* contains a subsequence that can be extended to a w^*-null sequence in X^*.

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