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Geodesic Ptolemy spaces and fixed points

Rafa Espínola[a], Adriana Nicolae[b,*]

[a] Departamento de Análisis Matemático, Universidad de Sevilla, Apdo. 1160, 41080 Sevilla, Spain

[b] Department of Mathematics, Babeş-Bolyai University Cluj-Napoca, Kogălniceanu 1, 400084, Cluj-Napoca, Romania

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ABSTRACT

In this paper we study the regularity of geodesic Ptolemy spaces and apply our findings to metric fixed point theory. It is an open question whether such spaces with a continuous midpoint map are CAT(0) spaces. We prove that if a certain uniform continuity is imposed on such a midpoint map then these spaces, if complete, are reflexive (that is, the intersection of decreasing families of bounded closed and convex subsets is nonempty) and that bounded sequences have unique asymptotic centers. These properties will then be applied to yield a series of fixed point results specific to CAT(0) spaces.

1. Introduction

In a metric space (X, d), the Ptolemy inequality says that

$$d(x, y)d(z, p) \leq d(x, z)d(y, p) + d(x, p)d(y, z) \quad \text{for every } x, y, z, p \in X.$$

A metric space where the Ptolemy inequality holds is called a Ptolemy metric space. It is shown in [1] that a normed space is an inner product space if and only if it is a Ptolemy space. Also, for each normed space $(X, \|\cdot\|)$, there exists a constant $1 \leq C \leq 2$ such that

$$\|x - y\| \, \|z - p\| \leq C(\|x - z\| \, \|y - p\| + \|x - p\| \, \|y - z\|) \quad \text{for every } x, y, z, p \in X,$$

(see for instance [2]). The smallest value of C such that the above inequality is satisfied is called the Ptolemy constant of the space X. The recent paper [3] examines the relation between the Ptolemy constant and the geometry of the space with application to the metric fixed point theory.

Moving into the metric setting, Foertsch and Schroeder prove in [4] that the Ptolemy inequality holds in the context of boundaries of CAT(−1) spaces endowed with an appropriate metric. This property is then used to study the relation between Gromov hyperbolic spaces and CAT(−1) spaces (see [4] for details).

CAT(0) spaces are Ptolemy spaces (see for instance [5] for a justification), but a geodesic Ptolemy space is not necessarily CAT(0). In fact, Foertsch et al. give in [6] an example of a geodesic Ptolemy space which is not even uniquely geodesic and thus not CAT(0). In the same paper the regularity of geodesic Ptolemy spaces is studied. In particular, it is shown that a proper geodesic Ptolemy space is uniquely geodesic, where the properness condition may be replaced by the existence of a continuous midpoint map. The authors raise then the still open question of whether a proper geodesic Ptolemy space (or a geodesic Ptolemy space with a continuous midpoint map) is CAT(0). It is easy to see that CAT(0) spaces are Busemann convex but being Busemann convex is a weaker property than being CAT(0). However, in [6] a characterization of CAT(0) spaces is

* Corresponding author.
E-mail addresses: espinola@us.es (R. Espínola), anicolae@math.ubbcluj.ro (A. Nicolae).

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