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The sum of a maximal monotone operator of type (FPV) and a maximal monotone operator with full domain is maximal monotone

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ABSTRACT

The most important open problem in Monotone Operator Theory concerns the maximal monotonicity of the sum of two maximal monotone operators provided that Rockafellar's constraint qualification holds.

In this paper, we prove the maximal monotonicity of $A+B$ provided that A and B are maximal monotone operators such that $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$, $A + N_{\overline{\operatorname{dom} B}}$ is of type (FPV), and $\operatorname{dom} A \cap \overline{\operatorname{dom} B} \subseteq \operatorname{dom} B$. The proof utilizes the Fitzpatrick function in an essential way.

1. Introduction

Throughout this paper, we assume that X is a real Banach space with norm $\|\cdot\|$, that X^* is the continuous dual of X, and that X and X^* are paired by $\langle\cdot,\cdot\rangle$. Let $A\colon X \rightrightarrows X^*$ be a *set-valued operator* (also known as multifunction) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A = \{(x,x^*) \in X \times X^* \mid x^* \in Ax\}$ be the *graph* of A. Recall that A is *monotone* if

$$\langle x-y, x^*-y^*\rangle \geq 0, \quad \forall (x,x^*) \in \operatorname{gra} A\ \forall (y,y^*) \in \operatorname{gra} A, \tag{1}$$

and *maximal monotone* if A is monotone and A has no proper monotone extension (in the sense of graph inclusion). Let $A : X \rightrightarrows X^*$ be monotone and $(x,x^*) \in X \times X^*$. We say (x,x^*) is *monotonically related to* $\operatorname{gra} A$ if

$$\langle x-y, y-y^*\rangle \geq 0, \quad \forall (y,y^*) \in \operatorname{gra} A.$$

Let $A : X \rightrightarrows X^*$ be maximal monotone. We say A is *of type (FPV)* if for every open convex set $U \subseteq X$ such that $U \cap \operatorname{dom} A \neq \varnothing$, the implication

$$x \in U \text{ and } (x,x^*) \text{ is monotonically related to } \operatorname{gra} A \cap U \times X^* \Rightarrow (x,x^*) \in \operatorname{gra} A$$

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