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A_p-weight and integrability of solutions in obstacle problems

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ABSTRACT

Integrability of solutions to obstacle problems is obtained under assumption that the obstacles belong to the A_p-weight class. This result is used to prove an existence result for A_p-weight problems.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ and let $H^1(\Omega)$ be the first-order Sobolev space of functions $u \in L^2(\Omega)$ whose distributional gradient ∇u belongs to $L^2(\Omega)$. If $\theta \in H^1(\Omega)$ and ψ is any function in Ω with values in $\mathbb{R} \bigcup \{-\infty, \infty\}$, then we write

$$L(v) = \int_\Omega |\nabla v|^2 \mathrm{d}x \tag{1.1}$$

and

$$\mathcal{K}_{\theta,\psi}(\Omega) = \{v \in H^1(\Omega), v|_{\partial\Omega} = \theta, v \geq \psi\}. \tag{1.2}$$

Obstacle problems naturally appear in the nonlinear potential theory and its applications include the study of fluid filtration in porous media, constrained heating, elasto-plasticity, optimal control, and financial math. It arises when one considers the shape taken by a soap film in a domain $\Omega \subset \mathbb{R}^n$ whose boundary position is fixed (see Plateau's problem), with the added constraint that the membrane is constrained to lie above some obstacle $\psi(x)$ in the interior of the domain as well. In this case, the energy functional to be minimized is the surface area integral

$$J(v) = \int_\Omega \sqrt{1 + |\nabla v|^2} \mathrm{d}x.$$

This problem can be linearized in the case of small perturbations by expanding the energy functional in terms of its Taylor series and taking the first term only, in which case the energy to be minimized is the standard Dirichlet energy (1.1) in $\Omega \subset \mathbb{R}^n$ where the functions v satisfy Dirichlet boundary conditions, and are in addition constrained to be greater than some given obstacle function ψ, that is, to find a function u that minimizes $L(v)$ in $K_{\theta,\psi}(\Omega)$.

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Since $\mathcal{K}_{\theta,\psi}(\Omega)$ is a closed convex set, there is a unique function u that minimizes $L(v)$ over all functions v belonging to $\mathcal{K}_{\theta,\psi}(\Omega)$, which is usually called the solution to the obstacle problem (see [1,2] for the existence).

Regularity and integrability of solutions to obstacle problems were widely considered, here we just refer to [3–13]. For example, it was shown in [7] that: let u be the solution of (1.1), (1.2) and $\theta, \psi \in C^2(\overline{\Omega})$, then there exist $\tau = \tau(N, p) > 0$, $\alpha = \alpha(N, p) > 0$, $\rho = \rho(\Omega)$ and $C_1 = C(\Omega, N, p, R)$, $C_2 = C(\Omega, N, p)$ with $R \in (0, 1)$ such that

$$\|u\|_{C^{1,\tau}(\Omega_R)} \le C(\|\theta\|_{C^2(\Omega)} + \|\psi\|_{C^2(\Omega)}) \tag{1.3}$$

and

$$\|u - \theta\|_{C^{1,\tau}(\Gamma_\rho)} \le C(\|\theta\|_{C^2(\Omega)} + \|\psi\|_{C^2(\Omega)}) \tag{1.4}$$

where $\Omega_R = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > R\}$ and $\Gamma_\rho = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \rho\}$. That is, $u \in C^{1,\alpha}(\overline{\Omega})$ if $\theta, \psi \in C^2(\overline{\Omega})$.

In this paper we will consider $K_{\theta,\psi}(\Omega)$-obstacle problem on A_p-weight property. Recall that a *weight* is a locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere. A weight w is said to be of *class* A_p $(1 < p < \infty)$ in $\mathbb{R}^n$ if it satisfies the condition

$$[w]_{A_p} = \sup_{\text{B balls in } \mathbb{R}^N} \left(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B w^{-\frac{1}{p-1}}\mathrm{d}x\right)^{p-1} < \infty. \tag{1.5}$$

$[w]_{A_p}$ is called the A_p-constant(or characteristic or norm) of weight w. A weight w is said to be of *class* $A_p(\Omega)$ $(1 < p < \infty)$ if

$$[w]_{A_p(\Omega)} = \sup_{\text{B balls in } \Omega} \left(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B w^{-\frac{1}{p-1}}\mathrm{d}x\right)^{p-1} < \infty. \tag{1.6}$$

$[w]_{A_p(\Omega)}$ is called the A_p-constant of weight w on $\Omega \subset \mathbb{R}^n$.

Muckenhoupt showed in [14] that the weights satisfying the A_p condition are exactly the weights for which the Hardy–Littlewood maximal function

$$Mf(x) = \sup_{x \in \text{B balls}} \frac{1}{|B|}\int_B |f(y)|\mathrm{d}y$$

is bounded on $L^p(w)$. Besides that it was shown in [16] that translations, isotropic dilations, and scalar multiples of A_p-weights are also A_p weights with the same A_p characteristic. An A_p-weight function has stability integrability, doubling measure and reverse Hölder inequality properties. See more results about A_p-weight in [15–22].

A natural question is: how can we find a nontrivial A_p-weight? To the best of our knowledge, the existence of a A_p-weight function has been discussed in very few papers. We will partially answer this question in this following.

In this paper we address two questions associated with the $K_{\theta,\psi}(\Omega)$-obstacle and A_p-weight problem. The first question is an existence problem for A_p-weight on a bounded domain. We show that under some natural assumptions a solution to the $K_{\theta,\psi}(\Omega)$-obstacle problem belongs to an $A_p(\Omega)$-weight class. This result can then be used to study the following whole space A_p-weight existence problem. If we have A_p-weight function ψ on a bounded interval $[a, b] \subset \mathbb{R}^1$, i.e. $\psi \in A_p([a, b])$, can we find an A_p-weight function $\varphi \in A_p(\mathbb{R}^1)$ or other ones?

Stability and higher integrability of derivatives of solutions have been studied in [6,23] for obstacle problems, but for A_p-weight obstacle problems a different approach must be employed. On the other hand, the existence of another A_p-weight function has been shown in [16], which depends heavily on the exponents p and hence is different from ours. From the definitions (1.5), (1.6), one can see that an A_p-weight is naturally an $A_p(\Omega)$-weight, but how about the inverse? It was shown in [24] that when a weight w belongs to $A_p(J_k)$ on all its finite many finite measure intervals J_k with $\Omega = \bigcup_{k=0}^m J_k$, then w belongs to $A_p(\Omega)$. Unfortunately, we have only one finite measure interval $[a, b]$ here and as it was remarked in [24] that $\Omega = \mathbb{R}$ is not trivial (see also the counterexample there). Hence, more care must be taken to get a A_p-weight on $\mathbb{R}$ from an interval $[a, b] \subset \mathbb{R}$.

From now on, u is a solution to (1.1) and (1.2) means that u is a function in $\mathcal{K}_{\theta,\psi}$ that minimizes $L(v)$ over all functions v belonging to $\mathcal{K}_{\theta,\psi}$. We will assume that $\theta, \psi \in C^2(\overline{\Omega})$ to avoid complications.

More regularity results about the solutions of $\mathcal{K}_{\theta,\psi}(\Omega)$-obstacle problem can be read as:

Lemma 1.1. *Let u be the solution of* (1.1), (1.2), *then*

(a) *u stays between $\lambda_1 = \min\theta(\partial\Omega)$ and $\lambda_2 = \max(\theta(\partial\Omega), \psi(\Omega))$*
(b) *u is superharmonic, and $spt(\Delta u) \subset \{u = \psi\}$.*

Proof. See Lemma 1 in [25]. □

Lemma 1.2. *Up to $C^{1,1}$, u is as regular as ψ. More precisely,*

(a) *Assume that ψ has a modulus of continuity $\sigma(r)$, then u has modulus of continuity $C\sigma(2r)$.*
(b) *Assume now that $\nabla\psi$ has modulus of continuity $\sigma(r)$, then ∇u has modulus of continuity $C\sigma(2r)$.*

Proof. See Lemma 2.3 in [7]. □

A direct result reads as the following.

Theorem 1.1. *Let $\psi(x)$ be of $A_p(\Omega)$ for some $1 < p < \infty$. Suppose that $\min_{\partial\Omega}\theta(x) = C > 0$, then the solution u to (1.1), (1.2) must be of $A_p(\Omega)$.*

Proof. Let $C = \min_{\partial\Omega}\theta(x) > 0$. Since $\psi \in C^2(\overline{\Omega})$, we have that $M = \max_{x\in\Omega}\psi(x) < \infty$.

From Lemma 1.1, we have that $0 < C \le u \le \max(C, M) < \infty$, so

$$\begin{aligned}\sup_{B \text{ balls in } \Omega}\left(\frac{1}{|B|}\int_B u(x)\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B u^{-\frac{1}{p-1}}\mathrm{d}x\right)^{p-1} &\le \max\{1, M/C\}\\ &< \infty. \quad \Box\end{aligned}$$

The rest of the article is organized as follows. In Section 2 we collect a number of auxiliary facts and prove for $\mathcal{K}_{\theta,\psi}(\Omega)$-obstacle problem about A_p-weight on $\Omega \subset \mathbb{R}^1$. In Section 3, we consider the problems on $\mathbb{R}^n$ for a radial case.

2. A_p-weight on $\mathbb{R}^1$

Let $\Omega = [a, b]$ be an interval in $\mathbb{R}$. Without loss of generality, we assume that $\Omega = [-1, 1]$. And, we have the following definition of $A_p(\Omega)$-weight in $\mathbb{R}^1$.

Definition 2.1. For $1 < p < \infty$, a function $w : [a, b] \to R_0^+$ is called an $A_p([a, b])$-*weight* iff

$$K_w = \sup_{I\subset[a,b]}\left(\frac{1}{|I|}\int_I w(x)\mathrm{d}x\right)\left(\frac{1}{|I|}\int_I w(x)^{-\frac{1}{p-1}}\mathrm{d}x\right) < \infty,$$

where I denotes an arbitrary subinterval of $[a, b]$.

At first, we will show that u is positive in Ω, that is

Lemma 2.1. *Suppose that u is a solution of the obstacle problem (1.1), (1.2) in $\Omega \subset \mathbb{R}^1$ with $\psi \in A_p(\Omega)$, then $u > 0$ in Ω.*

Proof. Since $\psi \in C^2(\overline{\Omega})$ and $\psi \in A_p(\Omega)$ with $\Omega = [-1, 1]$, there exists a point $\bar{x} \in [-1, 1]$, such that $\psi(\bar{x}) = \max_{x\in[-1,1]}\psi(x) > 0$. From Lemma 1.1, we know that u is superharmonic. By the weak maximum principle, it is not less than the harmonic function f on $[-1, \bar{x}]$ with $f(-1) = u(-1) \ge 0$ and $f(\bar{x}) = u(\bar{x}) \ge \psi(\bar{x}) > 0$ which is positive on $[-1, \bar{x}]$, as is u on $[-1, \bar{x}]$. Similarly, we have that u is positive on $[\bar{x}, 1]$, so u is positive on $\Omega = [-1, 1]$. $\Box$

See [24] for the following pasting A_p-weight lemma.

Lemma 2.2. *Let Ω be an open interval on $\mathbb{R}^1$, μ a Borel measure on Ω with spt $\mu = \Omega$. Assume that there exist some open intervals $J_0, J_1, \ldots, J_m$ such that*

(a) $\Omega = \bigcup_{k=0}^m J_k$;
(b) $J_0, J_1, \ldots, J_{m-1}$;
(c) $w \in A_{p,\mu}(J_k)$, *for every* $k = 0, 1, \ldots, m$;
(d) $w \ne 0$ *on* J_k, *for every* $k = 0, 1, \ldots, m-1$.

Then, $w \in A_{p,\mu}(\Omega)$.

Our result about $A_p(\Omega)$-weight on $\mathbb{R}^1$ can be read as:

Theorem 2.1. *Let u be the solution of obstacle problem (1.1), (1.2) on $\Omega \subset \mathbb{R}^1$ with $\psi \in A_p(\Omega)$, then $u \in A_p(\Omega)$.*

Proof. For $b = 1 \in \partial\Omega$. If $\theta(1) > 0$, it is done near the point $b = 1$ from Lemma 1.1. For the case of $\theta(1) = 0$, we have that

$$0 \le \psi(1) \le u(1) = \theta(1) = 0.$$

By the regularity results of Lemma 1.2, $u \in C^1(\Omega)$. Since $u \ge \psi$ on Ω, there exists an

$$\varepsilon_1 = \max\{\varepsilon \in (0, 2); u \equiv \psi \text{ in } (1-\varepsilon, 1)\}$$

or

$$\varepsilon_2 = \max\{\varepsilon \in (0, 2); u > \psi \text{ in } (1-\varepsilon, 1)\}.$$

Suppose the former case does, that is, there exists an $\varepsilon_1 \in (0, 2)$, such that $u \equiv \psi$ in $(1-\varepsilon_1, 1)$, then $u \in A_p([1-\varepsilon_1, 1])$.

Suppose the later case does, then $u_{xx} = 0$ on $[1-\varepsilon_2, 1]$ by Lemma 1.1(b) and $u(1-\varepsilon_2) > 0$ by Lemma 2.1. Hence we have that

$$u(x) = k_1 x - 1$$

for $x \in [1-\varepsilon_2, 1]$ with some $k_1 < 0$, and it is obvious that $u \in A_p[1-\varepsilon_2, 1]$.

The same applies for u at $\{x = -1\}$, that is, either $u \equiv \psi$ or $u(x) = k_2 x + 1$ with some $k_2 > 0$ near the point $\{x = 1\}$, so there exists an $\bar{\varepsilon}_2 \in (0, 2)$, such that $u \in A_p[-1, -1 + \bar{\varepsilon}_2]$ or there exists an $\bar{\varepsilon}_1 \in (0, 2)$, such that $u \in A_p[-1, -1 + \bar{\varepsilon}_1]$.

From the above, we have that: $u \in A_p[-1, -1 + \varepsilon_1]$ and $u \in A_p[1 - \varepsilon_2, 1]$. Note that for any given $\varepsilon \in (0, 1)$, we have that $u \in A_p[-1 + \varepsilon, 1 - \varepsilon]$ from Lemma 2.1. Then Lemma 2.2 implies that $u \in A_p[-1, 1]$, i.e. $u \in A_p(\Omega)$. $\square$

In the following, we will show that if an $A_p([a, b])$-weight ψ is given on an interval $[a, b]$, we can get an A_p-weight $\widetilde{\psi}$ on $\mathbb{R}$.

To guarantee that, we need the following result:

Lemma 2.3. *Let $1 < p < \infty$ and let $w_1 : [-a, 0] \to [0, \infty)$ and $w_2 : [0, a] \to [0, \infty)$ be A_p-weights. Define*

$$w(x) = \begin{cases} w_1(x), & \text{if } x \in [-a, 0], \\ w_2(x), & \text{if } x \in [0, a]. \end{cases}$$

Then $w : [-a, a] \to [0, \infty)$ is an $A_p[a, b]$-weight iff

$$0 < \liminf_{\varepsilon \to 0} \frac{\int_0^\varepsilon w(x)\mathrm{d}x}{\int_\varepsilon^0 w(x)\mathrm{d}x} < \limsup_{\varepsilon \to 0} \frac{\int_0^\varepsilon w(x)\mathrm{d}x}{\int_\varepsilon^0 w(x)\mathrm{d}x} < \infty.$$

Proof. See Theorem 3 in [26]. $\square$

Now, we can prove that

Theorem 2.2. *For any given $\psi \in A_p([0, 1])$ and any interval $[a, b] \subset \mathbb{R}^1$, we can get at least one nontrivial weight u, such that $u \in A_p([a, b])$.*

Proof. At first, we extend the definition domain of ψ from $[0, 1]$ to the whole $\mathbb{R}^1$ by defining

$$\psi(x) = \psi(2k - x)$$

for $x \in [k, 2k]$ with $k \in \mathbb{N}$ and $x \in [2k, k]$ with $k \in \mathbb{Z}^-$.

Then we give two ways to get $u \in A_p([a, b])$ for any given $[a, b] \subset \mathbb{R}^1$. The two solutions, u, may be different.

The first way is to take a large enough $m \in \mathbb{N}$ such that $[a, b] \subset [-m, m]$ and some nonnegative function θ, such that $\theta(-m) \geq 0$ and $\theta(m) \geq 0$. Let u be the obstacle problem $K_{\psi_m,\theta}([-m, m])$, where ψ_m is the restriction of ψ on $[-m, m]$, then $\psi \in A_p([-m, m])$ by Lemma 2.2 and $u \in A_p([-m, m])$ by Theorem 2.1. Then $u \in A_p([a, b])$ since $[u]_{A_p[a,b]} \leq [u]_{A_p[-m,m]} < \infty$.

The second way is to define u_k to be the minimizer for the obstacle problem on $[k, k + 1]$ for $\psi \in A_p([k, k + 1])$ and $\theta(k) \geq 0$ at each $k \in \mathbb{N}$. Then each $u_k \in A_p([k, k + 1])$ by Theorem 2.1 and we can paste finitely many u_k to get $u \in A_p([-m, m])$ for any given $m \in \mathbb{N}$ such that $[a, b] \subset [-m, m]$ by Lemma 2.2. Then, we also have a $u \in A_p([a, b])$. $\square$

Remark 2.1. From the proof of Theorem 2.2, one can see that:

(a) Since the choice of θ is arbitrarily, by taking different θ, we can get different $u \in A_p([a, b])$ even if ψ is given.
(b) In fact, we get a $u \in A_p(\mathbb{R})$ in the latter procedure. It is worth mentioning that in Lemma 2.2, $\Omega \subset \mathbb{R}$ is bounded and there are counterexamples to the weights pasting onto $\mathbb{R}$ (see Remark 7 in [24]).

3. A_p-weight on $\mathbb{R}^n$ for radial case

In this section, we consider A_p-weight on $\Omega = B_1(0) \subset \mathbb{R}^n$. In this case, we assume that the obstacles θ and ψ are radial symmetry, that is, $\psi(x) = \psi(|x|)$.

At first, we collect some facts for the solution u to the obstacle problem.

Lemma 3.1. *Let u be the solution of the obstacle problem (1.1), (1.2) on $B_1(0)$ with θ and ψ are radial symmetry, then u must be radial symmetry.*

Proof. Since θ and ψ are radial symmetry, for any operator A on $\mathbb{R}^n$ with $|Ax| = |x|$ for any $x \in \mathbb{R}^n$, we get new functions ψ_A and θ_A defined by

$$\psi_A(x) = \psi(Ax) = \psi(|Ax|) = \psi(|x|) = \psi(x)$$

and $\theta_A(x) = \theta(Ax) = \theta(x)$ for any $x \in \mathbb{R}^n$.

Let u_A be the solution of (1.1) on K_{ψ_A,θ_A}, it is obvious that

$$u_A(x) = u(Ax), \tag{3.1}$$

where $u(Ax)$ is the solution of (1.1) with obstacle $\psi(Ax)$ and $\theta(Ax)$.

Since $K_{\psi_A,\theta_A} \equiv K_{\psi,\theta}$ is a convex set, by the uniqueness of the solution,

$$u_A(x) = u(x). \tag{3.2}$$

Now, (3.1) and (3.2) give that $u(Ax) = u(x)$ for any operator A with $|Ax| = |x|$. By the arbitrary nature of A, we have that u is radial symmetry. □

Lemma 3.2. *Suppose that u is a solution of the obstacle problem (1.1), (1.2) in $B_1(0) \subset \mathbb{R}^n$ with $\psi \in A_p(B_1(0))$ and θ, ψ are radial symmetry, then $u > 0$ in Ω.*

Proof. $\psi \in A_p(B_1(0))$ implies that it is positive almost everywhere in $B_1(0)$, so there exists a point $\bar{x} \in B_{1/2}(0)$, such that $\psi(\bar{x}) > 0$. Since u is superharmonic, by the weak maximum principle, it is not less than the harmonic function f on $B_1(0) \setminus B_{|\bar{x}|}(0)$ with $f(x) = u(|\bar{x}|) \geq 0$ for any $x \in \mathbb{R}^N$ with $|x| = |\bar{x}|$ and $f(x) = u(x) \geq 0$ for any $x \in \mathbb{R}^N$ with $|x| = 1$, which is positive on $B_1(0) \setminus B_{|\bar{x}|}(0)$, so we have

$$u > 0 \quad \text{in } B_1(0) \setminus B_{|\bar{x}|}(0). \tag{3.3}$$

Since u is the minimizer for (1.1) on $K_{\psi,\theta}(B_1(0))$, it must be the minimizer for (1.1) on $K_{\psi,\theta}(B_{|\bar{x}|}(0))$. Then, by Lemma 1.1(a) with $\overline{\theta}(x) = u(x)$ for any $|x| = |\bar{x}|$, it is obvious that

$$u(x) \geq u(\bar{x}) = u(|\bar{x}|) > 0 \tag{3.4}$$

for any $x \in B_{|\bar{x}|}(0)$.

Combining (3.3) and (3.4) we get that u is positive everywhere in $B_1(0)$. □

Theorem 3.1. *Let u be the solution of the obstacle problem (1.1), (1.2) on $B_1(0)$ with $\psi \in A_p(B_1(0))$ for $p > 2$ and ψ, θ being radial symmetry, we have that $u \in A_p(B_1(0))$.*

Proof. Suppose that $\theta(x) = \theta(|x|) > 0$ for any $x \in B_1(0)$. By Lemma 1.1, there exist $C_1, C_2 > 0$ such that $0 < C_1 \leq u(x) \leq C_2 < \infty$, then it is obvious that $u \in A_p(B_1(0))$.

For the case of $\theta(x) = 0$ for $|x| = 1$, since ψ, θ and u are all radial symmetry and $u(x) = \theta(x) = \psi(x) = 0$ for $|x| = 1$, there is an $\varepsilon_1 > 0$ or an $\varepsilon_2 > 0$ as in the proof of Theorem 2.1 such that $u \equiv \psi$ for all $x \in B_1(0)$ with $|x| \in [1-\varepsilon_1, 1]$ or $\psi < u$ for all $x \in B_1(0)$ with $|x| \in (1-\varepsilon_2, 1)$.

Case I: There exists an $\varepsilon_1 > 0$ such that $u \equiv \psi$ for $|x| \in [1-\varepsilon_1, 1]$. Then, for any $B_r(y) \subset B_1(0)$.

(i) If $r \geq \varepsilon_1/4$, we have that

$$\begin{aligned} |B_r(y)|^{-p} \left(\int_{B_r(y)} u \mathrm{d}x \right) \left(\int_{B_r(y)} u^{-\frac{1}{p-1}} \mathrm{d}x \right)^{p-1} &\leq |B_{\varepsilon_1/4}|^{-p} \left(\int_{B_r(0)} u \mathrm{d}x \right) \left(\int_{B_r(0)} u^{-\frac{1}{p-1}} \mathrm{d}x \right)^{p-1} \\ &\leq C \left(\int_{B_r(0)} u^{-\frac{1}{p-1}} \mathrm{d}x \right)^{p-1}, \end{aligned} \tag{3.5}$$

where we have used the results of Lemma 1.1 that u is bounded above by λ_2, which depends only on ψ and θ. Hence, the constant C in (3.5) is independent of r. Then, it is enough to show the integral on the right of the last inequality in (3.5) is bounded.

Note that u is superharmonic, it must larger than the harmonic function f on $B_1(0)$ with $f(1-\varepsilon_1) = u(1-\varepsilon_1) = \psi(1-\varepsilon_1) \geq 0$ and $f(1) = u(1) = \psi(1) = 0$ which has the type of $f(x) = C(|x|^{2-n} - 1) \geq 0$ for $n \geq 3$ and $f(x) = C(1-|x|) \geq 0$ for $n = 2$ when $|x| \in [1-\varepsilon_1, 1]$. We will consider only the case of $n \geq 3$ in the following, that is,

$$u(x) \geq C(|x|^{2-n} - 1) \geq 0 \tag{3.6}$$

for $|x| \in (1-\varepsilon_1, 1)$.

Since $\psi \in A_p(B_1(0))$, it is positive almost everywhere in $B_1(0)$, one can choose some $\tau_2 \in (0, \varepsilon_1)$ small enough, such that $\psi(1-\tau_2) > 0$ and

$$\begin{aligned} 1-(1-t)^{n-2} &= (n-2)t - \frac{(n-2)(n-3)}{2}(-t)^2 + \cdots - (-t)^{n-2} \\ &\geq \frac{n-2}{2} t \end{aligned} \tag{3.7}$$

for all $t \in (0, \tau_2)$.

Since u is the solution of (1.1) on $K_{\theta,\psi}(B_1(0))$, it must be the solution of (1.1) on $K_{u(1-\tau_2),\psi}(B_{1-\tau_2}(0))$, then by Lemma 1.1(a),

$$u(x) \geq \min u(\partial B_{1-\tau_2}(0)) = u(1-\tau_2) \geq \psi(1-\tau_2) > 0 \tag{3.8}$$

for all $x \in B_1(0)$ with $|x| \leq 1-\tau_2$.

From (3.6), (3.8), we have that

$$\begin{aligned}\left(\int_{B_r(0)} u^{-\frac{1}{p-1}}\mathrm{d}x\right)^{p-1} &= \left(\int_{B_{1-\tau_2}(0)} u^{-\frac{1}{p-1}}\mathrm{d}x + \int_{B_1(0)\setminus B_{1-\tau_2}(0)} u^{-\frac{1}{p-1}}\mathrm{d}x\right)^{p-1}\\ &\le \left(\int_{B_{1-\tau_2}(0)} \psi(1-\tau_2)^{-\frac{1}{p-1}}\mathrm{d}x + \int_{B_1(0)\setminus B_{1-\tau_2}(0)} u^{-\frac{1}{p-1}}\mathrm{d}x\right)^{p-1}\\ &\le \left(C + \int_{B_1\setminus B_{1-\tau_2}(0)} (|x|^{2-N}-1)^{-\frac{1}{p-1}}\mathrm{d}x\right)^{p-1}. \end{aligned}\tag{3.9}$$

And from (3.7), we have that

$$\begin{aligned}\int_{B_1\setminus B_{1-\tau_2}(0)} (|x|^{2-N}-1)^{-\frac{1}{p-1}}\mathrm{d}x &\le \int_{B_1\setminus B_{1-\tau_2}(0)} |x|^{\frac{N-2}{p-1}}(1-|x|^{N-2})^{-\frac{1}{p-1}}\mathrm{d}x\\ &\le C(N)\int_{1-\tau_2}^{1} r^{\frac{N-2}{p-1}+N-1}(1-r^{N-2})^{-\frac{1}{p-1}}\mathrm{d}r\\ &\le C\int_{1-\tau_2}^{1}[1-r^{N-2}]^{-\frac{1}{p-1}}\mathrm{d}r\\ &= C\int_0^{\tau_2}[1-(1-t)^{N-2}]^{-\frac{1}{p-1}}\mathrm{d}t\\ &\le C(N,p)\int_0^{\tau_2} t^{-\frac{1}{p-1}}\mathrm{d}t\\ &\le C<\infty \end{aligned}\tag{3.10}$$

for any $p > 2$.

Combining (3.5), (3.9) and (3.10), we have that for any $\varepsilon_1/4 \le r < 1$ and any $y \in B_1(0)$ such that $B_r(y) \subset B_1(0)$,

$$|B_r(y)|^{-p}\left(\int_{B_r(y)} u\mathrm{d}x\right)\left(\int_{B_r(y)} u^{-\frac{1}{p-1}}\mathrm{d}x\right)^{p-1} \le C < \infty. \tag{3.11}$$

(ii) If $0 < r \le \varepsilon_1/4$, then for $|y| \le 1-\varepsilon_1/2$, we have

$$B_r(y) \subset B_{1-\varepsilon_1/4}(0) \subset\subset B_1(0).$$

By Lemma 3.2, it is done. And for $|y| \ge 1-\varepsilon_1/2$, we have

$$\|u\|_{A_p(B_r(y))} = \|\psi\|_{A_p(B_r(y))} \le [\psi]_{A_p(B_1(0))} < \infty.$$

Case II: There exists and $\varepsilon_2 > 0$, such that $\psi < u$ for all $|x| \in (1-\varepsilon_2, 1)$.

(i) If $r \ge \varepsilon_2/4$, by Lemma 1.1, u must be harmonic on $B_1 \setminus B_{1-\varepsilon_2}(0)$ and has the type of $C(|x|^{2-n}-1)$ for $n \ge 3$ and $C(1-|x|)$ for $n = 2$. Then following the calculus just as in (i) of Case I.

(ii) If $0 < r < \varepsilon_2/4$, just follow the calculus as in (ii) in Case I.

From the above, we finally have that

$$u \in A_p(B_1(0)). \quad \square$$

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