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Second order initial value problems with non-absolute integrals in ordered Banach spaces

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ABSTRACT

In this paper we derive the existence and comparison results for second order initial value problems in ordered Banach spaces. The considered problems can be implicit, singular, functional, discontinuous and nonlocal. The main tools are fixed point results in ordered spaces and the theory of locally HL integrable vector-valued functions.

1. Introduction

In [1] a theory of Henstock–Lebesgue (HL) integrable vector-valued functions and fixed point results for mappings in partially ordered function spaces is applied to derive the existence and comparison results for the smallest and greatest solutions of first order initial value problems in an ordered Banach space E whose order cone is regular. Our purpose is now to study second order initial value problems. Similar problems are studied in [2] when E is a lattice-ordered Banach space, and in [3], where improper integrals are used. A novel feature in our study is that the right-hand sides of differential equations comprise locally HL integrable vector-valued functions. Recent results in the theory of these functions obtained in [4,5] allow us to apply fixed point results in ordered spaces derived in [6].

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The following special types are included in the considered problems:

- differential equations and initial conditions may be implicit;
- differential equations may be singular and nonlocal;
- differential equations may contain non-absolutely integrable functions;
- both the differential equations and the initial conditions may depend functionally on the unknown function and/or on its derivatives;
- both the differential equations and the initial conditions may contain discontinuous nonlinearities;
- problems on unbounded intervals;
- problems of random type.

When E is the sequence space c_0 we obtain results for infinite systems of initial value problems, as shown in examples. Moreover, concrete finite systems are solved by using symbolic programming.

2. Preliminaries

In this section we study HL integrability and differentiability of Banach space valued functions of real variables.

A function v from a compact real interval $[a, b]$ to a Banach space E is called HL *integrable* if there is a function $u : [a, b] \to E$, called a *primitive* of v, with the following property: To each $\epsilon > 0$ there corresponds such a function $\delta : [a, b] \to (0, \infty)$ that whenever $[a, b] = \cup_i [t_{i-1}, t_i]$ and $\xi_i \in [t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1, \ldots, m$, then

$$\sum_{i=1}^{m} \|u(t_i) - u(t_{i-1}) - v(\xi_i)(t_i - t_{i-1})\| < \epsilon. \tag{2.1}$$

Criteria for HL integrability which are sufficient in our applications are given by the following lemma.

Lemma 2.1 ([4, Lemma 1.12]). *Given a function $v : [a, b] \to E$, assume there exists a continuous function $u : [a, b] \to E$ and a countable subset Z of $[a, b]$ such that $u'(t) = v(t)$ for all $t \in [a, b] \setminus Z$. Then v is HL integrable on $[a, b]$, and u is a primitive of v.*

Remark 2.1. If the set Z in Lemma 2.1 is uncountable an extra condition (see Theorem 2.1) is needed to ensure HL integrability. Compared with Lebesgue and Bochner integrability, the definition of HL integrability is easier to understand because no measure theory is needed. Moreover, all Bochner integrable (in real-valued case Lebesgue integrable) functions are HL integrable, but not conversely. For instance, HL integrability encloses improper integrals. The real-valued function f defined on $[0, 1]$ by $f(0) = 0$ and $f(t) = t^2 \cos(1/t^2)$ is differentiable on $[0, 1]$, whence f' is HL integrable by Lemma 2.1. But f' is not Lebesgue integrable on $[0, 1]$. More generally, let t be called a singular point of the domain interval of a real-valued function being not Lebesgue integrable on any interval that contains t. Then (cf. [7]) there exist HL "integrable functions on an interval that admit a set of singular points with its measure as close as possible but not equal to that of the whole interval".

If v is HL integrable on $[a, b]$, it is HL integrable on every closed subinterval $[c, d]$ of $[a, b]$. The *Henstock–Kurzweil* integral of v over $[c, d]$ is defined by

$$^K\!\!\int_c^d v(s)\,\mathrm{d}s := u(d) - u(c), \quad \text{where } u \text{ is a primitive of } v.$$

The proofs for the results of the next lemma can be found, e.g., from [8].

Lemma 2.2. (a) *The a.e. equal functions are HL integrable and their integrals are equal if one of these functions is HL integrable.*
(b) *Every HL integrable function is strongly measurable.*
(c) *A Bochner integrable function $u : [a, b] \to E$ is HL integrable, and*

$$\int_J u(s)\,\mathrm{d}s = {}^K\!\!\int_J u(s)\,\mathrm{d}s \quad \text{whenever } J \text{ is a closed subinterval of } [a, b].$$

The set $\mathrm{HL}([a, b], E)$ of all HL integrable functions $u : [a, b] \to E$ is a vector space with respect to the usual addition and scalar multiplication of functions. Identifying a.e. equal functions it follows that the space $L^1([a, b], E)$ of Bochner integrable functions from $[a, b]$ to E is a subset of $\mathrm{HL}([a, b], E)$.

We say that a function u from a compact real interval J to a Banach space E satisfies the *Strong Lusin Condition* if for each $\epsilon > 0$ and for each null set Z of J there exists such a function $\delta : J \to (0, \infty)$ that $\sum_{i=1}^{n} \|u(s_{2i}) - u(s_{2i-1})\| < \epsilon$ if the sequence $(s_i)_{i=1}^{2n}$ of J is increasing, $\{\xi_i\}_{i=1}^{n} \subseteq Z$ and $\xi_i - \delta(\xi_i) < s_{2i-1} \le \xi_i \le s_{2i} < \xi_i + \delta(\xi_i)$ for every $i = 1, \ldots, n$.

A function $v : J \to E$ is said to be *a.e. differentiable*, if the derivative

$$v'(t) = \lim_{h \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t}$$

exists for a.e. $t \in J$.

The following version of the *Fundamental Theorem of Calculus* is proved in [4, Theorem 9.18]. It is equivalent to that presented without proof in [9, Theorem 2.4].

Theorem 2.1. *Given* $u, v : J \to E$ *and* $(t_0, x_0) \in J \times E$, *then the following conditions are equivalent.*

(a) *u satisfies the Strong Lusin Condition,* $u'(t) = v(t)$ *for a.e.* $t \in J$ *and* $u(t_0) = x_0$.
(b) v *is HL-integrable and* $u(t) = x_0 + {}^K\!\int_{t_0}^t v(s)\mathrm{d}s$ *for all* $t \in J$ *(so u is a primitive of* v*).*

If $u : J \to E$ is a.e. differentiable, define $u'(t) = 0$ in those points $t \in J$ where u is not differentiable.

The next result is a consequence of Theorem 2.1.

Corollary 2.1. *If* $u : J \to E$ *is a.e. differentiable, then u satisfies the Strong Lusin Condition on J if and only if* u' *is HL-integrable, and*

$$u(t) - u(t_0) = {}^K\!\int_{t_0}^t u'(s)\mathrm{d}s \quad \text{for all } t_0, t \in J.$$

Recall that a closed subset E_+ of a Banach space E is an *order cone* if $E_+ + E_+ \subseteq E_+$, $E_+ \cap (-E_+) = \{0\}$ and $cE_+ \subseteq E_+$ for each $c \geq 0$. It is easy to see that the order relation $\leq$, defined by

$$x \leq y \quad \text{if and only if} \quad y - x \in E_+,$$

is a partial ordering in E, and that $E_+ = \{y \in E : 0 \leq y\}$. The space E, equipped with this partial ordering, is called an *ordered Banach space*. The order cone E_+ is called *regular* if all increasing and order bounded sequences of E_+ converge.

In what follows we assume that E is an ordered Banach space with a regular order cone. Given an open or half-open interval J of $\mathbb{R}$, denote by $\mathrm{HL_{loc}}(J, E)$ the space of all strongly measurable functions $u : J \to E$ which are *locally HL integrable on J*, i.e., HL integrable on each compact subinterval of J. We assume that $\mathrm{HL_{loc}}(J, E)$ is ordered a.e. pointwise, i.e.

$$u \leq v \quad \text{if and only if} \quad u(s) \leq v(s) \text{ for a.e. } s \in J. \tag{2.2}$$

The following result plays a central role both in the theory and in applications of HL integrability in ordered Banach spaces. It is an easy consequence of [4, Lemma 9.11].

Lemma 2.3. *Assume that* $v_1, v_2 \in \mathrm{HL_{loc}}(J, E)$, *and that* $v_1 \leq v_2$. *Then*

$${}^K\!\int_a^t v_1(s)\,\mathrm{d}s \leq {}^K\!\int_a^t v_2(s)\,\mathrm{d}s \quad \text{for all } a, t \in J,\ a \leq t.$$

The next result follows from [5, Proposition 2.1].

Lemma 2.4. *Let E be a Banach space ordered by a regular order cone. If* $v : J \to E$ *is strongly measurable, if* $u_\pm \in \mathrm{HL_{loc}}(J, E)$, *and if* $u_- \leq v \leq u_+$, *then* $v \in \mathrm{HL_{loc}}(J, E)$.

In our study the following result is indispensable. It is adopted from [5, Proposition 3.2]. As for its more detailed proof, see [4, Chapter 9].

Lemma 2.5. *Let W be a nonempty set in an order interval* $[w_-, w_+]$ *of* $\mathrm{HL_{loc}}(J, E)$.

(a) *If W is well-ordered (every nonempty subset of W has the smallest element), it contains an increasing sequence which converges a.e. pointwise to* $\sup W$.
(b) *If W is inversely well-ordered (every nonempty subset of W has the greatest element), it contains a decreasing sequence which converges a.e. pointwise to* $\inf W$.

The following fixed point result is a consequence of [10, Theorem A.2.1], or [6, Theorem 1.2.1 and Proposition 1.2.1]. Together with Lemma 2.5 it forms the main tool used to prove our main results.

Lemma 2.6. *Given a partially ordered set* $P = (P, \leq)$, *and its order interval* $[w_-, w_+] = \{w \in P : w_- \leq u \leq w_+\}$, *assume that* $G : [w_-, w_+] \to [w_-, w_+]$ *is increasing, i.e.,* $Gu \leq Gv$ *whenever* $w_- \leq u \leq v \leq w_+$, *and that each well-ordered chain of the range* $G[P]$ *of G has a supremum in P and each inversely well-ordered chain of* $G[P]$ *has an infimum in P. Then G has the smallest and greatest fixed points, and they are increasing with respect to G.*

3. Second order initial value problems

We shall study in this section the second order initial value problem

$$\begin{cases} Lu(t) := \dfrac{\mathrm{d}}{\mathrm{d}t}(p(t)u'(t)) = f(t, u, u', Lu) \quad \text{for a.e. } t \in J := (a, b), \\ \lim\limits_{t \to a+} (p(t)u'(t)) = c(u, u', Lu), \qquad \lim\limits_{t \to a+} u(t) = d(u, u', Lu), \end{cases} \tag{3.1}$$

where $-\infty < a < b \le \infty$, $f : J \times \mathrm{HL}_{\mathrm{loc}}(J, E)^3 \to E$, $c, d : \mathrm{HL}_{\mathrm{loc}}(J, E)^3 \to E$, and $p : J \to \mathbb{R}_+$. We are looking for the smallest and greatest solutions of (3.1) from the set

$$Y := \{u : J \to E : u \text{ and } pu' \text{ satisfy the Strong Lusin Condition and are a.e. differentiable on compact intervals of } J\}. \tag{3.2}$$

The IVP (3.1) can be converted to a system of equations which does not contain derivatives.

Lemma 3.1. *Assume that $\frac{1}{p} \in L^1_{\mathrm{loc}}([a, b), \mathbb{R}_+)$, and that $f(\cdot, u, v, w) \in \mathrm{HL}_{\mathrm{loc}}([a, b), E)$ for all $u, v, w \in \mathrm{HL}_{\mathrm{loc}}(J, E)$. Then u is a solution of the IVP (3.1) in Y if and only if $(u, u', Lu) = (u, v, w)$, where $(u, v, w) \in \mathrm{HL}_{\mathrm{loc}}(J, E)^3$ is a solution of the system*

$$\begin{cases} u(t) = d(u, v, w) + {}^K\!\displaystyle\int_a^t v(s)\,\mathrm{d}s, \quad t \in J, \\ v(t) = \dfrac{1}{p(t)}\left(c(u, v, w) + {}^K\!\displaystyle\int_a^t w(s)\,\mathrm{d}s\right), \quad t \in J, \\ w(t) = f(t, u, v, w) \quad \text{for a.e. } t \in J. \end{cases} \tag{3.3}$$

Proof. Assume that u is a solution of (3.1) in Y, and denote

$$w(t) = Lu(t) = \frac{\mathrm{d}}{\mathrm{d}t}(p(t)v(t)), \qquad v(t) = u'(t), \quad t \in J. \tag{3.4}$$

The differential equation, the initial conditions of (3.1), the definition (3.2) of Y and the notations (3.4) ensure by Corollary 2.1 that the third equation of (3.3) is satisfied, and that

$$\begin{aligned} {}^K\!\int_a^t w(s)\,\mathrm{d}s &= \lim_{r\to a+} {}^K\!\int_r^t w(s)\,\mathrm{d}s = \lim_{r\to a+} {}^K\!\int_r^t \frac{\mathrm{d}}{\mathrm{d}s}(p(s)v(s))\mathrm{d}s \\ &= \lim_{r\to a+}(p(t)v(t) - p(r)v(r)) = p(t)v(t) - c(u, v, w), \quad t \in J, \end{aligned}$$

and

$$\begin{aligned} u(t) - d(u, v, w) &= \lim_{r\to a+}(u(t) - u(r)) = \lim_{r\to a+} {}^K\!\int_r^t u'(s)\,\mathrm{d}s \\ &= {}^K\!\int_a^t u'(s)\,\mathrm{d}s = {}^K\!\int_a^t v(s)\,\mathrm{d}s, \quad t \in J. \end{aligned}$$

Thus the first and second equations of (3.3) hold.

Conversely, let (u, v, w) be a solution of the system (3.3) in $\mathrm{HL}_{\mathrm{loc}}(J, E)^3$. The first equation of (3.3) implies by Theorem 2.1 that $v = u'$, that u is a.e. differentiable and satisfies the Strong Lusin Condition on every closed interval of J, and that the second initial condition of (3.1) is fulfilled. Since $v = u'$, it follows from the second equation of (3.3) that

$$p(t)u'(t) = c(u, u', w) + {}^K\!\int_a^t w(s)\,\mathrm{d}s, \quad t \in J. \tag{3.5}$$

By Theorem 2.1 the Eq. (3.5) implies that $p \cdot u'$ is a.e. differentiable and satisfies the Strong Lusin Condition on every closed interval of J, and thus $u \in Y$, as well as that

$$w(t) = \frac{\mathrm{d}}{\mathrm{d}t}(p(t)u'(t)) = Lu(t) \quad \text{for a.e. } t \in J. \tag{3.6}$$

The Eqs. (3.5) and (3.6) imply that the first initial condition of (3.1) holds. The validity of the differential equation of (3.1) is a consequence of the third equation of (3.3), the Eq. (3.6), and the fact that $v = u'$. □

Assume that $\mathrm{HL}_{\mathrm{loc}}(J, E)$ and $\mathrm{HL}_{\mathrm{loc}}([a, b), E)$ are ordered a.e. pointwise, that Y is ordered pointwise, and that the functions p, f, c and d satisfy the following hypotheses:

(p0) $\frac{1}{p} \in L^1_{\mathrm{loc}}([a, b), \mathbb{R}_+)$.

(f0) $f(\cdot, u, v, w)$ is strongly measurable, and there exist such $h_-, h_+ \in \mathrm{HL}_{\mathrm{loc}}([a, b), E)$ that $h_- \le f(\cdot, u, v, w) \le h_+$ for all $u, v, w \in \mathrm{HL}_{\mathrm{loc}}(J, E)$.

(f1) There exists a $\lambda \ge 0$ such that $f(\cdot, u_1, v_1, w_1) + \lambda w_1 \le f(\cdot, u_2, v_2, w_2) + \lambda w_2$ whenever $u_i, v_i, w_i \in \mathrm{HL}_{\mathrm{loc}}(J, E)$, $i = 1, 2$, $u_1 \le u_2$, $v_1 \le v_2$ and $w_1 \le w_2$.

(c0) $c_\pm \in E$, and $c_- \le c(u_1, v_1, w_1) \le c(u_2, v_2, w_2) \le c_+$ if $u_i, v_i, w_i \in \mathrm{HL}_{\mathrm{loc}}(J, E)$, $i = 1, 2$, $u_1 \le u_2$, $v_1 \le v_2$ and $w_1 \le w_2$.

(d0) $d_\pm \in E$, and $d_- \le d(u_1, v_1, w_1) \le d(u_2, v_2, w_2) \le d_+$ if $u_i, v_i, w_i \in \mathrm{HL}_{\mathrm{loc}}(J, E)$, $i = 1, 2$, $u_1 \le u_2$, $v_1 \le v_2$ and $w_1 \le w_2$.

Our main existence and comparison result for the IVP (3.1) reads as follows.

Theorem 3.1. *Assume that the hypotheses* (p0), (f0), (f1), (c0) *and* (d0) *hold. Then the IVP* (3.1) *has the smallest and greatest solutions in Y, and they are increasing with respect to f, c and d.*

Proof. Assume that $P = \mathrm{HL}_{\mathrm{loc}}(J, E)^3$ is ordered componentwise. We shall first show that the vector-functions x_+, x_- given by

$$x_\pm(t) := \begin{pmatrix} d_\pm + {}^K\!\!\int_a^t \frac{1}{p(s)}\left(c_\pm + {}^K\!\!\int_a^s h_\pm(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ \frac{1}{p(t)}\left(c_\pm + {}^K\!\!\int_a^t h_\pm(s)\,\mathrm{d}s\right) \\ h_\pm(t) \end{pmatrix} \tag{3.7}$$

define functions $x_\pm \in P$. The third components of $x_\pm$ belong to $\mathrm{HL}_{\mathrm{loc}}(J, E)$ by the hypothesis (f0). Since $1/p$ is locally Lebesgue integrable and the functions $t \mapsto c_\pm + {}^K\!\int_a^t h_\pm(s)\,\mathrm{d}s$ are continuous on $[a, b)$, then the second components of $x_\pm$ are strongly measurable by [6, Theorem 1.4.3]. Moreover, if $t_1 \in J$ then for each $t \in [a, t_1]$, $\left\|\frac{1}{p(t)}\left(c_\pm + {}^K\!\int_a^t h_\pm(s)\,\mathrm{d}s\right)\right\| \le M_\pm \frac{1}{p(t)}$, where $M_\pm = \max\left\{\left\|c_\pm + {}^K\!\int_a^t h_\pm(s)\,\mathrm{d}s\right\| : t \in [a, t_1]\right\}$. Thus the second components of $x_\pm$ are locally Bochner integrable, and belong to $\mathrm{HL}_{\mathrm{loc}}(J, E)$. This result implies that the first components of $x_\pm$ are defined and continuous, whence they belong to $\mathrm{HL}_{\mathrm{loc}}(J, E)$.

Similarly, by applying also the given hypotheses and the results of Theorem 2.1 and Lemmas 2.3 and 2.4 one can verify that the relations

$$\begin{cases} G_1(u, v, w)(t) := d(u, v, w) + {}^K\!\!\int_a^t v(s)\,\mathrm{d}s, \quad t \in J, \\ G_2(u, v, w)(t) := \frac{1}{p(t)}\left(c(u, v, w) + {}^K\!\!\int_a^t w(s)\,\mathrm{d}s\right), \quad t \in J, \\ G_3(u, v, w)(t) := \frac{f(t, u, v, w) + \lambda w(t)}{1 + \lambda}, \quad t \in J, \end{cases} \tag{3.8}$$

define an increasing mapping $G = (G_1, G_2, G_3) : [x_-, x_+] \to [x_-, x_+]$.

Let W be a well-ordered chain in the range of G. The sets $W_1 = \{u : (u, v, w) \in W\}$, $W_2 = \{v : (u, v, w) \in W\}$ and $W_3 = \{w : (u, v, w) \in W\}$ are well-ordered and order-bounded chains in $\mathrm{HL}_{\mathrm{loc}}(J, E)$. It then follows from Lemma 2.5 that the supremums of W_1, W_2 and W_3 exist in $\mathrm{HL}_{\mathrm{loc}}(J, E)$. Obviously, $(\sup W_1, \sup W_2, \sup W_3)$ is the supremum of W in P. Similarly one can show that each inversely well-ordered chain of the range of G has the infimum in P.

The above proof shows that the operator $G = (G_1, G_2, G_3)$ defined by (3.8) satisfies the hypotheses of Lemma 2.6, and therefore G has the smallest fixed point $x_* = (u_*, v_*, w_*)$ and the greatest fixed point $x^* = (u^*, v^*, w^*)$. It follows from (3.8) that (u_*, v_*, w_*) and (u^*, v^*, w^*) are solutions of the system (3.3). According to Lemma 3.1, u_* and u^* belong to Y and are solutions of the IVP (3.1).

To prove that u_* and u^* are the smallest and greatest of all solutions of (3.1) in Y, let $u \in Y$ be any solution of (3.1). In view of Lemma 3.1, $(u, v, w) = (u, u', Lu)$ is a solution of the system (3.3). Applying the hypotheses (f0), (c0) and (d0) it is easy to show that $x = (u, v, w) \in [x_-, x_+]$, where $x_\pm$ are defined by (3.7). Thus $x = (u, v, w)$ is a fixed point of $G = (G_1, G_2, G_3) : [x_-, x_+] \to [x_-, x_+]$, defined by (3.8). Because $x_* = (u_*, v_*, w_*)$ and $x^* = (u^*, v^*, w^*)$ are the smallest and greatest fixed points of G, then $(u_*, v_*, w_*) \le (u, v, w) \le (u^*, v^*, w^*)$. In particular, $u_* \le u \le u^*$, whence u_* and u^* are the smallest and greatest of all solutions of the IVP (3.1).

The last assertion is an easy consequence of the last conclusion of Lemmas 2.4 and 2.6 and the definition (3.8) of $G = (G_1, G_2, G_3)$. □

As a special case we obtain an existence result for the IVP

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(p(t)u'(t)) = g\left(t, u(t), u'(t), \frac{\mathrm{d}}{\mathrm{d}t}(p(t)u'(t))\right) \quad \text{for a.e. } t \in J, \\ \lim_{t\to a+}(p(t)u'(t)) = c, \qquad \lim_{t\to a+} u(t) = d. \end{cases} \tag{3.9}$$

Corollary 3.1. *Assume hypothesis* (p0), *and let $g : J \times E \times E \times E \to E$ satisfy the following hypotheses:*

(g0) *$g(\cdot, u(\cdot), v(\cdot), w(\cdot))$ is strongly measurable and $h_- \le g(\cdot, u(\cdot), v(\cdot), w(\cdot)) \le h_+$ for all $u, v, w \in \mathrm{HL}_{\mathrm{loc}}(J, E)$ and for some $h_\pm \in \mathrm{HL}_{\mathrm{loc}}([a, b), E)$.*

(g1) *There exists a $\lambda \ge 0$ such that $g(t, x_1, x_2, x_3) + \lambda x_3 \le g(t, y_1, y_2, y_3) + \lambda y_3$ for a.e. $t \in J$ and whenever $x_i \le y_i$ in E, $i = 1, 2, 3$.*

Then the IVP (3.9) has for each choice of $c, d \in E$ the smallest and greatest solutions in Y. Moreover, these solutions are increasing with respect to g, c and d.

Proof. If $c, d \in E$, the IVP (3.9) is reduced to (3.1) when we define

$$f(t,u,v,w)=g(t,u(t),v(t),w(t)),\quad t\in J,\ u,v,w\in \mathrm{HL}_{\mathrm{loc}}(J,E),$$
$$c(u,v,w)\equiv c,\qquad d(u,v,w)\equiv d,\quad u,v,w\in \mathrm{HL}_{\mathrm{loc}}(J,E).$$

The hypotheses (g0) and (g1) imply that f satisfies the hypotheses (f0) and (f1). The hypotheses (c0) and (d0) are also valid, whence we conclude that (3.1), with f, c and d defined above, and hence also (3.9), has the smallest and greatest solutions due to Theorem 3.1. The last assertion follows from the last assertion of Theorem 3.1. □

In the next example $[x]$ denotes the greatest integer $\le x \in \mathbb{R}$.

Example 3.1. Determine the smallest and greatest solutions of the following system of implicit singular IVP's in $J = (0,\infty)$

$$\begin{cases} L_1u_1(t) := \dfrac{\mathrm{d}}{\mathrm{d}t}(\sqrt{t}u_1'(t)) = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(t\sin\dfrac{1}{t}\right) + \dfrac{\left[\int_1^2(u_2(s)+u_2'(s)+L_2u_2(s))\,\mathrm{d}s\right]}{1+\left|\left[\int_1^2(u_2(s)+u_2'(s)+L_2u_2(s))\,\mathrm{d}s\right]\right|},\\ L_2u_2(t) := \dfrac{\mathrm{d}}{\mathrm{d}t}(\sqrt{t}u_2'(t)) = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(t\cos\dfrac{1}{t}\right) + \dfrac{\left[\int_1^2(u_1(s)+u_1'(s)+L_1u_1(s))\,\mathrm{d}s\right]}{1+\left|\left[\int_1^2(u_1(s)+u_1'(s)+L_1u_1(s))\,\mathrm{d}s\right]\right|},\\ \lim\limits_{t\to0+}\sqrt{t}u_1'(t) = \dfrac{[u_2'(1)]}{1+|[u_2'(1)]|},\qquad \lim\limits_{t\to0+}u_1(t)=\dfrac{[u_2(1)]}{1+|[u_2(1)]|},\\ \lim\limits_{t\to0+}\sqrt{t}u_2'(t) = \dfrac{[u_1'(1)]}{1+|[u_1'(1)]|},\qquad \lim\limits_{t\to0+}u_2(t)=\dfrac{[u_1(1)]}{1+|[u_1(1)]|}. \end{cases} \tag{3.10}$$

Solution: System (3.10) is a special case of (3.1) by setting $E=\mathbb{R}^2$, $a=0$, $b=\infty$, $p(t)=\sqrt{t}$, and $f=(f_1,f_2)$, c, d given by

$$\begin{cases} f_1(t,(u_1,u_2),(v_1,v_2),(w_1,w_2)) = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(t\sin\dfrac{1}{t}\right) + \dfrac{\left[\int_1^2(u_2(s)+v_2(s)+w_2(s))\,\mathrm{d}s\right]}{1+\left|\left[\int_1^2(u_2(s)+v_2(s)+w_2(s))\,\mathrm{d}s\right]\right|},\\ f_2(t,(u_1,u_2),(v_1,v_2),(w_1,w_2)) = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(t\cos\dfrac{1}{t}\right) + \dfrac{\left[\int_1^2(u_1(s)+v_1(s)+w_1(s))\,\mathrm{d}s\right]}{1+\left|\left[\int_1^2(u_1(s)+v_1(s)+w_1(s))\,\mathrm{d}s\right]\right|},\\ c((u_1,u_2),(v_1,v_2),(w_1,w_2)) = \left(\dfrac{[v_2(1)]}{1+|[v_2(1)]|},\dfrac{[v_1(1)]}{1+|[v_1(1)]|}\right),\\ d((u_1,u_2),(v_1,v_2),(w_1,w_2)) = \left(\dfrac{[u_2(1)]}{1+|[u_2(1)]|},\dfrac{[u_1(1)]}{1+|[u_1(1)]|}\right). \end{cases} \tag{3.11}$$

In view of Lemmas 2.1 and 2.4 the hypotheses (f0), (f1), (c0) and (d0) hold when $h_\pm(t)=\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(t\sin\frac{1}{t}\right)\pm1,\ \frac{\mathrm{d}}{\mathrm{d}t}\left(t\cos\frac{1}{t}\right)\pm1\right)$, $\lambda=0$ and $c_\pm=d_\pm=(\pm1,\pm1)$. Thus (3.10) has the smallest and greatest solutions. The functions x_- and x_+ defined by (3.7) can be calculated, and their first components are

$$u_-(t) = -1-\frac{2\sqrt{2\pi}}{3}-2\sqrt{t}+\frac{2t\sqrt{t}}{3}\sin\frac{1}{t}+\frac{4\sqrt{t}}{3}\cos\frac{1}{t}+\frac{4\sqrt{2\pi}}{3}\mathrm{FrC}\left(\sqrt{\frac{2}{\pi t}}\right)-\frac{2t\sqrt{t}}{3},$$

$$v_-(t) = -1-\frac{2\sqrt{2\pi}}{3}-2\sqrt{t}+\frac{2t\sqrt{t}}{3}\cos\frac{1}{t}-\frac{4\sqrt{t}}{3}\sin\frac{1}{t}+\frac{4\sqrt{2\pi}}{3}\mathrm{FrC}\left(\sqrt{\frac{2}{\pi t}}\right)-\frac{2t\sqrt{t}}{3},$$

$$u_+(t) = 1-\frac{2\sqrt{2\pi}}{3}+2\sqrt{t}+\frac{2t\sqrt{t}}{3}\sin\frac{1}{t}+\frac{4\sqrt{t}}{3}\cos\frac{1}{t}+\frac{4\sqrt{2\pi}}{3}\mathrm{FrC}\left(\sqrt{\frac{2}{\pi t}}\right)+\frac{2t\sqrt{t}}{3},$$

$$v_+(t) = 1-\frac{2\sqrt{2\pi}}{3}+2\sqrt{t}+\frac{2t\sqrt{t}}{3}\cos\frac{1}{t}-\frac{4\sqrt{t}}{3}\sin\frac{1}{t}+\frac{4\sqrt{2\pi}}{3}\mathrm{FrC}\left(\sqrt{\frac{2}{\pi t}}\right)+\frac{2t\sqrt{t}}{3},$$

where

$$\mathrm{FrC}(x)=\int_0^x\cos\left(\frac{\pi}{2}t^2\right)\,\mathrm{d}t$$

is the *Fresnel cosine integral*. According to Lemma 3.1 the smallest solution of (3.10) is equal to the first component of the smallest fixed point of $G = (G_1, G_2, G_3)$, defined by (3.8), with f, c and d given by (3.11) and $p(t) = \sqrt{t}$. Calculating the iterations $G^n x_-$ it turns out that $G^4 x_- = G^5 x_-$, whence $G_1^4 x_-$ is the smallest solution of (3.10). Similarly, one can show that $G_1^2 x_+$ is the greatest solution of (3.10). The exact expressions of the components of these solutions are

$$u_{1*}(t) = -\frac{3}{4} - \frac{2\sqrt{2\pi}}{3} - \sqrt{t} + \frac{2t\sqrt{t}}{3}\sin\frac{1}{t} + \frac{4\sqrt{t}}{3}\cos\frac{1}{t} + \frac{4\sqrt{2\pi}}{3}\mathrm{FrC}\left(\sqrt{\frac{2}{\pi t}}\right) - \frac{t\sqrt{t}}{2},$$

$$u_{2*}(t) = -\frac{2}{3} - \frac{2\sqrt{2\pi}}{3} - \sqrt{t} + \frac{2t\sqrt{t}}{3}\cos\frac{1}{t} + \frac{4\sqrt{t}}{3}\sin\frac{1}{t} + \frac{4\sqrt{2\pi}}{3}\mathrm{FrC}\left(\sqrt{\frac{2}{\pi t}}\right) - \frac{8t\sqrt{t}}{15},$$

$$u_1^*(t) = \frac{2}{3} - \frac{2\sqrt{2\pi}}{3} + \frac{4\sqrt{t}}{3} + \frac{2t\sqrt{t}}{3}\sin\frac{1}{t} + \frac{4\sqrt{t}}{3}\cos\frac{1}{t} + \frac{4\sqrt{2\pi}}{3}\mathrm{FrC}\left(\sqrt{\frac{2}{\pi t}}\right) + \frac{16t\sqrt{t}}{27},$$

$$u_2^*(t) = \frac{3}{4} - \frac{2\sqrt{2\pi}}{3} + \frac{4\sqrt{t}}{3} + \frac{2t\sqrt{t}}{3}\cos\frac{1}{t} - \frac{4\sqrt{t}}{3}\sin\frac{1}{t} + \frac{4\sqrt{2\pi}}{3}\mathrm{FrC}\left(\sqrt{\frac{2}{\pi t}}\right) + \frac{7t\sqrt{t}}{12}.$$

Example 3.2. Let E be the space (c_0), ordered coordinatewise and normed by the sup-norm. By Lemma 2.1 the mappings $h_\pm : [0,\infty) \to c_0$, defined by $h_\pm(0) = (0, 0, \ldots)$ and

$$h_\pm(t) = \left(\frac{1}{nt}\sin\frac{1}{t} + \frac{1}{n}\cos\frac{1}{t} \pm \frac{1}{n}\right)_{n=1}^{\infty}, \quad t \in (0,\infty), \tag{3.12}$$

belong to $\mathrm{HL_{loc}}([0,\infty), E)$. Thus these mappings are possible upper and lower boundaries for f in the hypothesis (f0) of Theorem 3.1 and for g in the hypothesis (g0) of Corollary 3.1 when $E = c_0$. Choosing $c_\pm = (\pm n^{-1})_{n=1}^\infty$, $d_\pm = (\pm n^{-1})_{n=1}^\infty$ and $p(t) := \sqrt{t}$, the solutions of the initial value problems

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(\sqrt{t}u'(t)) = h_\pm(t) \quad \text{for a.e. } t \in (0,\infty), \\ \lim\limits_{t\to 0+}(\sqrt{t}u'(t)) = c_\pm, \qquad \lim\limits_{t\to 0+} u(t) = d_\pm \end{cases} \tag{3.13}$$

are

$$\begin{cases} u_+(t) = \left(\dfrac{1}{n}\left(\dfrac{2}{3}t\sqrt{t}\cos\dfrac{1}{t} - \dfrac{4}{3}\sin\dfrac{1}{t} + \dfrac{4\sqrt{2\pi}}{3}\mathrm{FrC}\left(\sqrt{\dfrac{2}{\pi t}}\right) + 2\sqrt{t} + \dfrac{2}{3}t\sqrt{t} + 1 - \dfrac{2}{3}\sqrt{2\pi}\right)\right)_{n=1}^{\infty}, \\ u_-(t) = \left(\dfrac{1}{n}\left(\dfrac{2}{3}t\sqrt{t}\cos\dfrac{1}{t} - \dfrac{4}{3}\sin\dfrac{1}{t} + \dfrac{4\sqrt{2\pi}}{3}\mathrm{FrC}\left(\sqrt{\dfrac{2}{\pi t}}\right) - 2\sqrt{t} - \dfrac{2}{3}t\sqrt{t} - 1 - \dfrac{2}{3}\sqrt{2\pi}\right)\right)_{n=1}^{\infty}. \end{cases} \tag{3.14}$$

Consider, in particular, the infinite system of initial value problems

$$\begin{cases} L_n u_n(t) := \dfrac{\mathrm{d}}{\mathrm{d}t}(\sqrt{t}u_n'(t)) = \dfrac{1}{n}\left(\dfrac{1}{t}\sin\dfrac{1}{t} + \cos\dfrac{1}{t} + f_n(u, u', Lu)\right) \quad \text{a.e. on } (0,\infty), \\ \lim\limits_{t\to 0+}(\sqrt{t}u_n'(t)) = \dfrac{c_n}{n}, \qquad \lim\limits_{t\to 0+} u_n(t) = \dfrac{d_n}{n}, \quad n \in \mathbb{N}. \end{cases} \tag{3.15}$$

Setting $u = (u_n)_{n=1}^\infty$, $Lu = (L_n u_n)_{n=1}^\infty$, and assuming that each $f_n : \mathrm{HL_{loc}}((0,\infty), c_0)^3 \to \mathbb{R}$ is increasing with respect to every argument, and $-1 \le c_n, d_n, f_n(u, v, w) \le 1$ for all $u, v, w \in \mathrm{HL_{loc}}((0,\infty), c_0)$ and $n \in \mathbb{N}$, then (3.15) has the smallest and greatest solutions $u_* = (u_{*n})_{n=1}^\infty$ and $u^* = (u_n^*)_{n=1}^\infty$, and they belong to the order interval $[u_-, u_+]$, where $u_\pm$ are given by (3.14).

Remark 3.1. Lemma 2.6 is an efficient tool to prove the existence and comparison results for solutions of various problems by their conversions to fixed point equations of the form $x = Gx$ in suitable ordered function spaces P. It is applied also in [1] to first order singular and functional initial value problems with $P = \mathrm{HL_{loc}}(J, E)^i$, $i = 1, 2$, and in [11] to second order singular and nonlocal boundary value problems in ordered Banach spaces with $P = \mathrm{HL}([a,b], E)^3$. Lemma 2.5, proved first in [5], and more detailed in [4], is the key for these applications when the considered problems contain HL integrable functions. Thanks to the use of these order theoretic lemmas, syntactically similar proofs yield in [11] and in this paper results that are syntactically similar but semantically completely different because of different problems, hypotheses, spaces P, and mappings G.

Examples of ordered Banach spaces whose order cones are regular are given, e.g., in [12, Section 2.2] and in [6, Section 5.8]. For instance, spaces $\mathbb{R}^m$, $m = 1, 2, \ldots$, ordered coordinatewise and normed by any norm, spaces l^p, $p \in [1, \infty)$, and c_0, ordered componentwise and normed by their usual norms, and spaces $L^p(\Omega, \mathbb{R})$, where $p \in [1, \infty)$ and $\Omega = (\Omega, \mathcal{A}, \mu)$ is a measure space, equipped with p-norm and a.e. pointwise ordering, have regular order cones. In particular, we can choose E to be one of these spaces in the above considerations.

Problems of the form (3.1), may include many kinds of special types. For instance, they can be

- singular, since $\lim_{t \to a+} p(t) = 0$ is allowed, and since $\lim_{t \to a+} f(t, u, v)$ and/or $\lim_{t \to b-} f(t, u, v)$ need not exist;
- nonlocal, because the functions c, d and f may depend functionally on u, u' and/or Lu;
- discontinuous, since the dependencies of c, d and f on u, u' and/or Lu can be discontinuous;
- problems on unbounded intervals, because the case $b = \infty$ is included in (3.1);
- finite systems when $E = \mathbb{R}^m$;
- infinite systems when E is l^p or c_0-space;
- of random type when $E = L^p(\Omega)$ and Ω is a probability space.

Problems belonging to types listed above when $E = \mathbb{R}$ are studied, e.g., in [13,10,14,15]. Initial value problems in ordered Banach spaces are studied, e.g., in [2,16,3,5,6].

The solutions of examples have been calculated by using simple Maple programming.

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