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Strongly convex functions of higher order

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ABSTRACT

The notion of strongly n-convex functions with modulus $c > 0$ is introduced and investigated. Relationships between such functions and n-convex functions in the sense of Popoviciu as well as generalized convex functions in the sense of Beckenbach are given. Characterizations by derivatives are presented. Some results on strongly Jensen n-convex functions are also given.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and c be a positive constant. A function $f : I \to \mathbb{R}$ is called *strongly convex with modulus c* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2, \tag{1}$$

for all $x, y \in I$ and $t \in [0, 1]$. Strongly convex functions have been introduced by Polyak [1] and they play an important role in optimization theory. Many properties of these functions can be found, among others, in [2–6].

In the classical theory of convex functions (i.e. functions satisfying (1) with $c = 0$) their natural generalization are convex functions of higher order. Let us recall the definition. Let $n \in \mathbb{N}$ and $x_0, \ldots, x_n$ be distinct points in I. Denote by $[x_0, \ldots, x_n; f]$ the divided difference of f at $x_0, \ldots, x_n$ defined by the recurrence

$$[x_0; f] = f(x_0),$$
$$[x_0, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}, \quad n \in \mathbb{N}.$$

Following Hopf [7] and Popoviciu [8] a function $f : I \to \mathbb{R}$ is called *convex of order n* (or *n-convex*) if

$$[x_0, \ldots, x_{n+1}; f] \geq 0$$

for all $x_0 < \cdots < x_{n+1}$ in I. It is well known (and easy to verify) that 1-convex functions are ordinary convex functions. Many results on n-convex functions one can found, among others, in [8,9,4,10–13]. In this paper we introduce the notion of strongly n-convex functions and investigate properties of this class of functions. Let c be a positive constant and $n \in \mathbb{N}$. We say that a function $f : I \to \mathbb{R}$ is *strongly convex of order n with modulus c* (or *strongly n-convex with modulus c*) if

$$[x_0, \ldots, x_{n+1}; f] \geq c, \tag{2}$$

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