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On the extension of the solutions of Hamilton–Jacobi equations

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ABSTRACT

We consider the viscosity solution of a homogeneous Dirichlet problem for the eikonal equation in a bounded set Ω. We suppose that the Hamiltonian, $H(x,p) = \langle A(x)p, p\rangle - 1$, is strictly convex w.r.t. the variables p and of class $C^{1,1}$ w.r.t. the variables x. Then the solution of the Dirichlet problem admits an extension to a neighbourhood of Ω, $\overline{u}$, such that $\overline{u}$ is still a viscosity solution of the eikonal equation if and only if $\partial\Omega$ satisfies an exterior sphere condition. The above result, in particular, provides a characterization of the boundary singularities and a regularity theorem (up to the boundary) for the solution of the eikonal equation.

1. Introduction and statement of the results

Let $\Omega \subset\subset \Omega_1 \subset\subset \mathbb{R}^n$ be two open bounded sets, i.e. Ω is an open bounded set and its closure is a subset of Ω_1. Let u be the (continuous) viscosity solution of the eikonal equation

$$\langle A(x)Du(x), Du(x)\rangle = 1 \quad x \in \Omega, \tag{1.1}$$

with the homogeneous Dirichlet boundary condition

$$u(x) = 0 \quad x \in \partial\Omega. \tag{1.2}$$

Here $A(\cdot)$ are $n \times n$ symmetric matrices, Du is the gradient of u, and $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. We recall that a continuous function, $u : \Omega \to \mathbb{R}$, is a **viscosity solution**[1] of (1.1) iff for every φ of class C^1 and every $x \in \Omega$ such that $u - \varphi$ has a local minimum at x we have

$$\langle A(x)D\varphi(x), D\varphi(x)\rangle = 1.$$

Set

$$H(x, p) = \langle A(x)p, p\rangle - 1.$$

We assume that $A(\cdot)$ is nondegenerate: there exists $\lambda \in]0, 1]$ such that

$$\lambda I \le A(x) \le \frac{1}{\lambda} I \quad \forall x \in \Omega_1. \tag{1.3}$$

Here I is the identity matrix. We say, for short, that H is $C^{1,1}$ nondegenerate if condition (1.3) holds and $A(x)$ is of class $C^{1,1}$ in Ω_1 w.r.t. the variables x. As usual, we denote by $B_r(x)$ the Euclidean open ball with radius r and center at x.

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[1] We point out that our definition of a viscosity solution is not the usual one for general first-order Hamilton–Jacobi equations. On the other hand, it is well-known (see e.g. [1]) that, in the case of Hamiltonians convex with respect to the gradient, the usual definition is equivalent to ours.

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