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# On the Nagumo uniqueness theorem

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ABSTRACT

By a convenient reparametrization of the integral curves of a nonlinear ordinary differential equation (ODE), we are able to improve the conclusions of a recent generalization of Nagumo's uniqueness theorem. In this way, we establish a flexible uniqueness criterion for ODEs without Lipschitz-like nonlinearities.



## 1. Introduction

Let us consider the nonlinear ordinary differential equation

$$x' + f(t,x) = 0, \quad t > 0, \tag{1}$$

where the nonlinearity $f$ is assumed continuous everywhere and $f(t,0) \equiv 0$. In this note, we shall discuss a set of very permissive conditions to be imposed on $f$ such that the only solution of (1) starting from $x(0) = 0$ will be the trivial solution.

Several results in the recent literature use such uniqueness criteria to perform detailed phase plane analysis of various differential equations and systems from the applied sciences, with an emphasis on fluid mechanics; see e.g., [1–5].

One of the most powerful uniqueness theorems concerning (1) is due to Nagumo [6] and states that, if $\lim_{t\searrow 0} f(t,x) = 0$ uniformly with respect to $x$ in $[-1,1]$ and

$$|f(t,x_1) - f(t,x_2)| \leq \frac{1}{t} \cdot |x_1 - x_2|, \quad t > 0,\ |x_1|, |x_2| \leq 1, \tag{2}$$

the initial value problem associated with the equation possesses a unique solution. The result has been generalized to $n$th-order equations in [7–9] and other papers.

In a different direction [10], given the smooth function $u : [0,1] \to [0,+\infty)$ with $u(0) = 0$ and $u'(t) > 0$ everywhere in $(0,1]$, if the nonlinearity $f$ verifies the restriction

$$|f(t,x_1) - f(t,x_2)| \leq \frac{u'(t)}{u(t)} \cdot |x_1 - x_2|, \quad t \in (0,1],\ |x_1|, |x_2| \leq 1, \tag{3}$$

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