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# Learning figures with the Hausdorff metric by fractals—towards computable binary classification

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**Abstract** We present learning of *figures*, nonempty compact sets in Euclidean space, based on *Gold's learning model* aiming at a *computable* foundation for binary classification of multivariate data. Encoding real vectors with no numerical error requires *infinite* sequences, resulting in a gap between each real vector and its *discretized* representation used for the actual machine learning process. Our motivation is to provide an analysis of machine learning problems that explicitly tackles this aspect which has been glossed over in the literature on binary classification as well as in other machine learning tasks such as regression and clustering. In this paper, we amalgamate two processes: discretization and binary classification. Each learning target, the set of real vectors classified as positive, is treated as a figure. A learning machine receives discretized vectors as input data and outputs a sequence of discrete representations of the target figure in the form of *self-similar sets*, known as *fractals*. The generalization error of each output is measured by the *Hausdorff metric*. Using this learning framework, we reveal a hierarchy of learnable classes under various learning criteria in the track of traditional analysis based on Gold's learning model, and show a mathematical connection between machine learning and fractal geometry by measuring the complexity of learning using the *Hausdorff dimension* and the *VC dimension*. Moreover, we

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