Int J Comput Vis (2013) 104:270–285
DOI 10.1007/s11263-013-0617-0

Combinatorial Optimization of the Discretized Multiphase Mumford–Shah Functional

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Received: 10 December 2011 / Accepted: 3 February 2013 / Published online: 3 March 2013

Abstract The Mumford–Shah model has been one of the most influential models in image segmentation and denoising. The optimization of the multiphase Mumford–Shah energy functional has been performed using level sets methods that optimize the Mumford–Shah energy by evolving the level sets via the gradient descent. These methods are very slow and prone to getting stuck in local optima due to the use of gradient descent. After the reformulation of the 2-phase Mumford–Shah functional on a graph, several groups investigated the hierarchical extension of the graph representation to multi class. The discrete hierarchical approaches are more effective than hierarchical (or direct) multiphase formulation using level sets. However, they provide approximate solutions and can diverge away from the optimal solution. In this paper, we present a discrete alternating optimization for the discretized Vese–Chan approximation of the piecewise constant multiphase Mumford–Shah functional that directly minimizes the multiphase functional without recursive bisection on the labels. Our approach handles the nonsubmodularity of the multiphase energy function and provides a global optimum if the image estimation data term is known apriori.

Keywords Multiphase Mumford–Shah · Discrete optimization · Image segmentation · Image denoising

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1 Introduction

The formulation of the image segmentation problem as an energy minimization problem has been one of the most powerful and commonly used techniques in the past couple of decades. For example, the piecewise constant Mumford–Shah segmentation aims at minimizing the integral of the intensity deviations around the mean of each class.

Minimizing such integral equations can be done in two ways (Bruckstein et al. 1997): First, to find the associated partial differential equation and use numerical methods to solve the PDE, in which case, discretization of the solution domain is used to optimize the continuous problem. An alternative method is to find the analogous discrete problem and employ combinatorial optimization tools to solve it. Numerical solution of PDE's via level sets method has been extensively used in image segmentation. However, it suffers from several drawbacks such as: (1) it requires tuning for many parameters. When different parameters are used to discretize the solution domain, the results may vastly change. (2) It depends on local optimization tools such as the gradient descent method for the optimization of the continuous problem, but these are known to get stuck in local optima and jeopardize the quality of the output.

Recent studies (El-Zehiry et al. 2011; Grady and Alvino 2009; Darbon and Sigelle 2005) demonstrate the favorability of the discrete approach. They unify two of the most important frameworks in image segmentation: graph representation and variational formulations. The ultimate goal of the new hybrid framework is to provide new discrete formulations that can benefit from the arsenal of variational formulations literature and meanwhile extend the robustness of these algorithms by employing combinatorial optimization algorithms that can capture global optima in a relatively very short time.